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IYPT International Young Physicists' Tournament

BEAD DYNAMICS DEMYSTIFIED

This talk shall be mostly theoretical, so don't expect demonstrations and/or experimental setups. We shall make use of the notations and some results for the case of a non-tilted axis and no friction ($\alpha = 0$ and $b = 0$) from the paper:

"The bead on a rotating hoop revisited: an unexpected resonance" L. Raviola *et al*, Eur. J. Phys. **38** (2017) 015005

1. The prototype example

A small bead of mass m can slide without friction on a circular hoop that is in a vertical plane and has a radius R_0. The hoop rotates at a constant rate ω about a vertical diameter. Find the angle θ at which the bead is in a vertical equilibrium.

$$T = \frac{1}{2} m R_0^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta), \quad V = -mgR_0 \cos\theta, \quad \mathcal{L}(\theta, \dot{\theta}) = T - V$$

$$U_{\text{eff}}(\theta) = mR_0 \left(-\frac{1}{2}\omega^2 R_0 \sin^2\theta - g\cos\theta\right)$$

$$U'_{\text{eff}}(\theta) = mR_0 \sin\theta\,(-\omega^2 R_0 \cos\theta + g)$$

Answer: If $\omega \le \omega_C \equiv \sqrt{\frac{g}{R_0}}$, then we have equilibrium positions $\theta = \theta^*$ for $\theta^* = 0$ (stable) and $\theta^* = \pi$ (unstable). In the case when $\omega > \omega_C$ both these equilibrium positions become unstable but we get two new stable ones at

$$\theta^* = \pm\left|\cos^{-1}\left(\frac{\omega_C}{\omega}\right)\right|$$

which break the discrete reflection symmetry of the equations.

Also: The frequency of oscillations around $\theta^* = 0$ is determined to be $\omega_0 = \sqrt{\omega_C^2 - \omega^2}$ for small ω.

Question: $U''_{\text{eff}}(\theta) = 0$ when $\omega = \omega_C$ and $\theta = 0$. Then, how can you argue that we have a stable equilibrium in this case?

Aside: symmetry breaking and other notions

Suppose we have some symmetry transformations (discrete or continuous) that leave the equations of motion invariant. Then the set of all solutions shall be invariant (as a set) under these transformations. Individual solutions may be or may not be invariant. In the latter case we may say that we have a loss of a symmetry or symmetry breaking.

It's intriguing that development of a new theory quite often comes with a different set of symmetry transformations (e.g. Galilean invariance is lost in the relativity theory, being replaced by the Lorenz/Poincare invariance; time-reversibility symmetry of an ensemble of finite number of particles is lost in the thermodynamics due to its second law.)

Examples:

- Press a steel needle (thin cylindrical rod) along the longest axis. At a certain moment the needle shall buckle (instead of being compressed further), thus breaking the rotational symmetry around the longest axis.
- Newton/Kepler problem is rotationally invariant but the stable elliptic trajectories of the planets break this symmetry as their semi-major axis (or Laplace-Runge-Lenz vector) has a definite direction. It's ironic that this apparent loss of symmetry is connected with the abundance of invariants in this problem.

Spontaneous symmetry breaking is a well-defined notion in the realm of Quantum field theory and it may only exist in systems with infinite degrees of freedom (i.e. fields). There are attempts to define analogous notion for the classical mechanics but quite often they employ concepts beyond classical mechanics.

Situation with the first/second order *phase transitions* analogies is somewhat similar. These transitions take place when the number of particles tends to infinity but some equations describing these phenomena look similar to the equations at hand.

Personally, I'm not in favour of the use of concepts having only superficial similarity with the situation in our problem.

Aside: bifurcations vs potentials

The notion of a (local) bifurcation could be well defined:

> "A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change." *Wikipedia*

and it is applicable for the problem in hand.

The question which is open for debate is whether it provides a really good insight into the physics of the problem. Bifurcation diagrams are fine but, personally, I'd find a plot of the (effective) potential as a function of ω to be a better aid for the physicists' intuition about the specifics of the problem.

Aside: Lagrangian approach

University textbooks and scholarly papers often employ the Lagrangian approach in order to obtain laws of motion. Having a Lagrangian function $\mathcal{L}(q_i, \dot{q}_i) = T - V$ i.e. the difference of the kinetic and potential energies, one can easily obtain (in the simplest case) the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Generalised coordinates q_i characterize the position of the dynamical system and their time derivatives $\dot{q}_i$ are the generalized velocities. Lagrangian equations have the same form independently of the choice of generalised coordinates. Additional terms may arise if there are non-conservative forces or constraints.

Exercise 1: Obtain the equations of motion for a pendulum in terms of the following generalised coordinates measuring the displacement from the equilibrium: vertical shift, horizontal shift, length of the arc traversed, deflection angle.

Suppose that the bob of the pendulum rotates around the vertical axis. To what extent shall we have similar equations or identical situation as in the prototype example?

2. IYPT problem specifics

- **FINITE SIZE BALL VS POINTLIKE BEAD**
- **GROOVE VS WIRE HOOP**
- **ROLLING VS SLIDING – HOW SHALL THIS AFFECT INERTIAL PROPERTIES OF THE BODY?**
- **FRICTION**

3. Equilibrium positions, stability

A natural first step would be to analyse which equilibrium positions shall survive, which shall alter their stability, which shall move somewhat? (E.g. shall $\theta^* = \pi$ still be in the list of equilibrium positions?)

4. Oscillations around stable equilibrium positions

One possible next step is to investigate the oscillations around the stable equilibrium positions. That's why we shall make a brief digression into some theoretical aspects of oscillations.

5. Small oscillations theory

Suppose we have a simple one-dimensional dynamical system (without drag or friction) of a point mass in a potential well, such that there is a stable equilibrium at the bottom. Let's choose a generalised coordinate q, such that it is zero at the bottom and let the Lagrangian have the form:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}M\dot{q}^2 - U(q)$$

In almost all cases $U(q)$ could be approximated near the bottom by a quadratic function, say

$$U(q) \approx \tfrac{1}{2}Kq^2 \text{ for small } q.$$

Then the Lagrangian $\mathcal{L}(q, \dot{q}) = \frac{1}{2}M\dot{q}^2 - \frac{1}{2}Kq^2$ yields equation of motion which describe harmonic oscillations with frequency $\upsilon = \sqrt{\frac{K}{M}}$ and this frequency shall be independent of the amplitude (as long as it remains small).

Exercise 2: Calculate the frequency of small oscillations for the case of **Exercise 1** and check whether we shall have equal frequencies for all generalised coordinates.

Exercise 3: Calculate the frequency of small oscillations around $\theta^* = \cos^{-1}\left(\frac{\omega_C}{\omega}\right)$ for the case of large ω in the prototype example.

This approximation provides a useful and simple method to tackle numerous engineering problems. When we have a dynamical system with N degrees of freedom around a stable equilibrium, then we have an even stronger result – the motion shall be a sum of N independent harmonic oscillations. This means that the effects of whatever coupling between the different degrees of freedom tend to disappear for small oscillations. (One should be warned that the coordinates along which we shall have independent harmonic oscillations may not be the coordinates we have chosen at the beginning but rather some linear combinations of them.)

6. Large oscillations theory

If we are interested in oscillations which are large enough and for which the quadratic approximation is no more adequate, then we still have an easy way to obtain the period of these oscillations. Starting from the equation for the energy of the system: $E = \frac{1}{2}M\dot{q}^2 + U(q)$ we obtain $\dot{q} = \sqrt{\frac{2(E-U(q))}{M}}$ and $dt = \sqrt{\frac{M}{2(E-U(q))}}dq$

Integrating from q_1 to q_2 – the turning points where $\dot{q} = 0$ we shall obtain the semiperiod of these oscillations as a function of the value of the energy E i.e.

$$T = \int_{q_1}^{q_2} \sqrt{\frac{2M}{(E - U(q))}}\,dq$$

This integral might not be solvable in elementary functions but one can always evaluate it numerically.

Exercise 4: Redo *Exercise 2* for large oscillations and compare the results.

7. What is the friction?

The prototype example was about an ideal case without drag or friction. The paper cited assumes a friction force proportional to the velocity after referring to: "Coulomb's law for rolling friction" R. Cross, Am. J. Phys. **84** (2016) 221

Do you agree that the arguments for such an assumption are strong enough? (One may find some papers which analyse a similar problem with dry friction.)

Or, maybe, you have to make an experiment in order to investigate what is the friction in your setup?

8. What can you do

Obviously, you can start with checking on your setup where the stable equilibrium positions are for different ω and what the frequencies of oscillations around them are. Compare the results with your predictions.

You may try to determine what the friction force is for your setup (and even try to investigate how it depends on velocity). Explore how oscillations slow down and whether this decay is compatible with the friction determined.

At the end, you may investigate what happens when we provide the ball with large initial velocity.

9. Sophistications

If you are more mathematically inclined, then you might like to try Lagrangian dynamics with constraints when tackling these topics:

- Rolling without sliding of the ball on the groove
- Reduction of dynamics on a sphere $\mathbb{S}^2$ (i.e. all possible positions of the ball) to a circle $\mathbb{S}^1$ (i.e. the positions on the hoop)

What shall change if we have a weak motor with constant torque (instead of ω = *const*), or constant angular momentum of the system *ball & hoop*?

10. Quiz

What's the most important difference between the IYPT problem and the prototype example from physicist's point of view?

How many dimensionless parameters we have here?

Do we have mass (in)dependence here?

What about equilibrium positions for $|\theta| > \frac{1}{2}\pi$?

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