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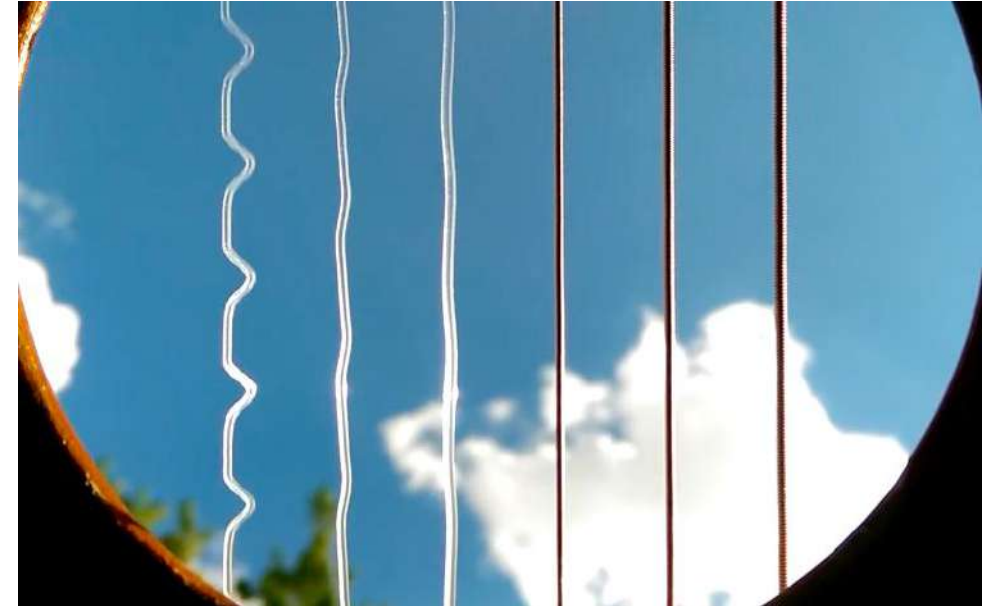
# Problem No. 11

## Guitar String

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Ronald Dobos

12. Nov 2020



Unrelated but nice phenomenon

Links to the videos and quotes from the papers in notes ↓

# Problem statement

- A **periodic** force is applied to a steel guitar string using an electromagnet. Investigate the **motion** of the guitar string **around its resonance frequency**
- Arising questions:
  - Need to be steel and guitar string? Why so?
  - How exactly forced?
  - What can be interesting about a vibrating string??

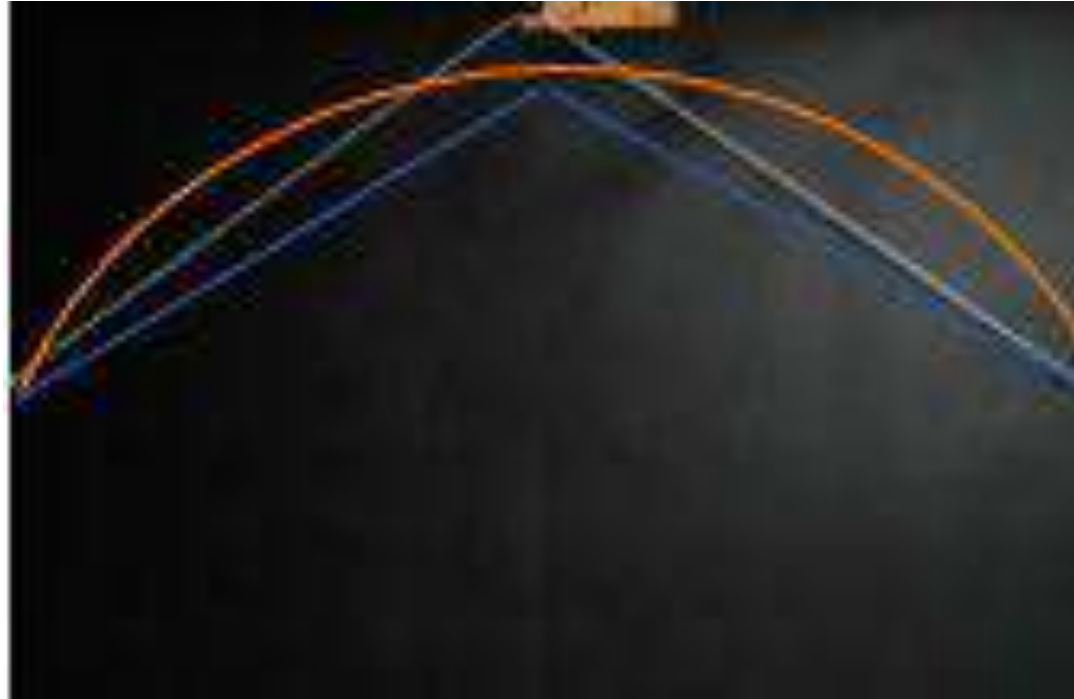
# Outline

- Building intuition
  - Go through simpler cases
- Review some basic theory
- Understand its limitations
  - Finding the interesting part
- Discuss setup
- What has been done?
- What you could do

# Building intuition

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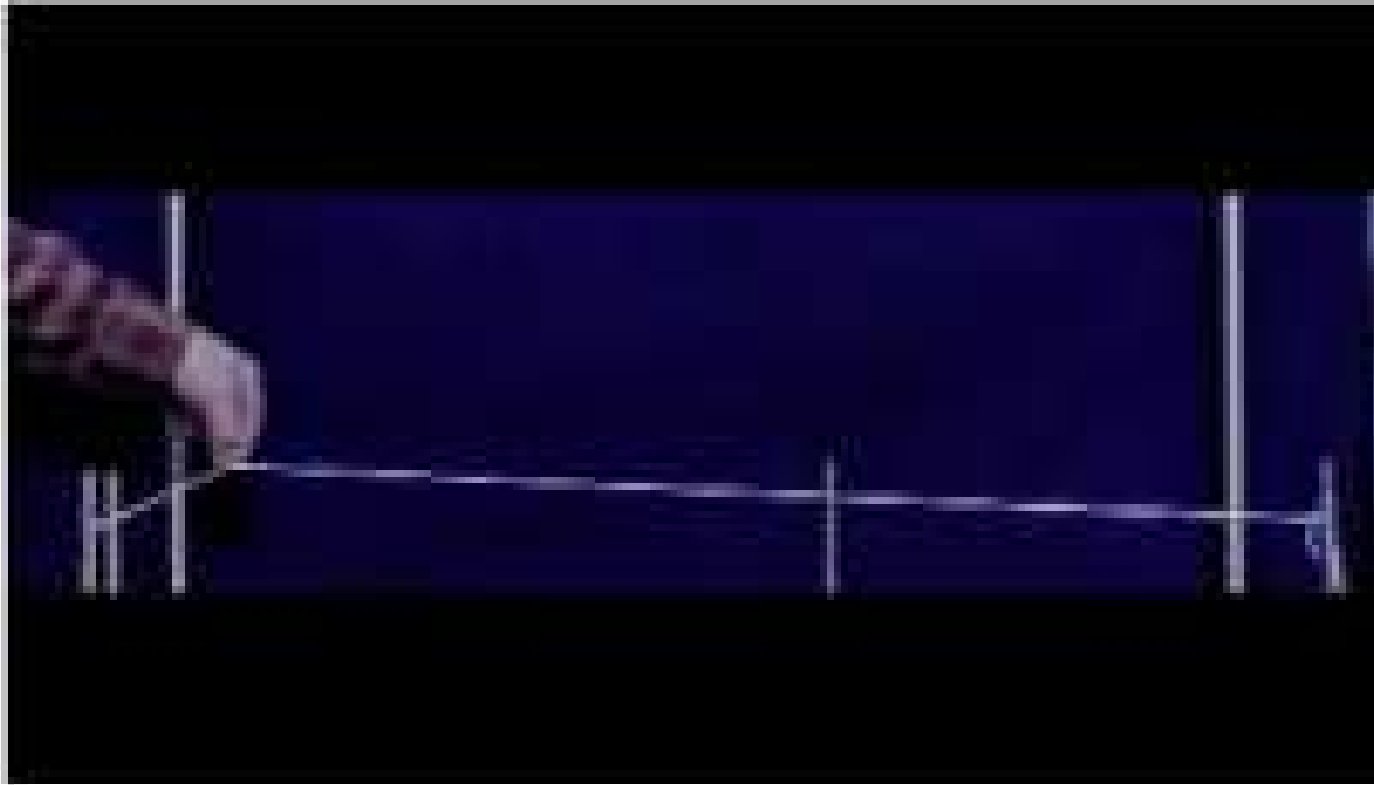
# Initial disturbance



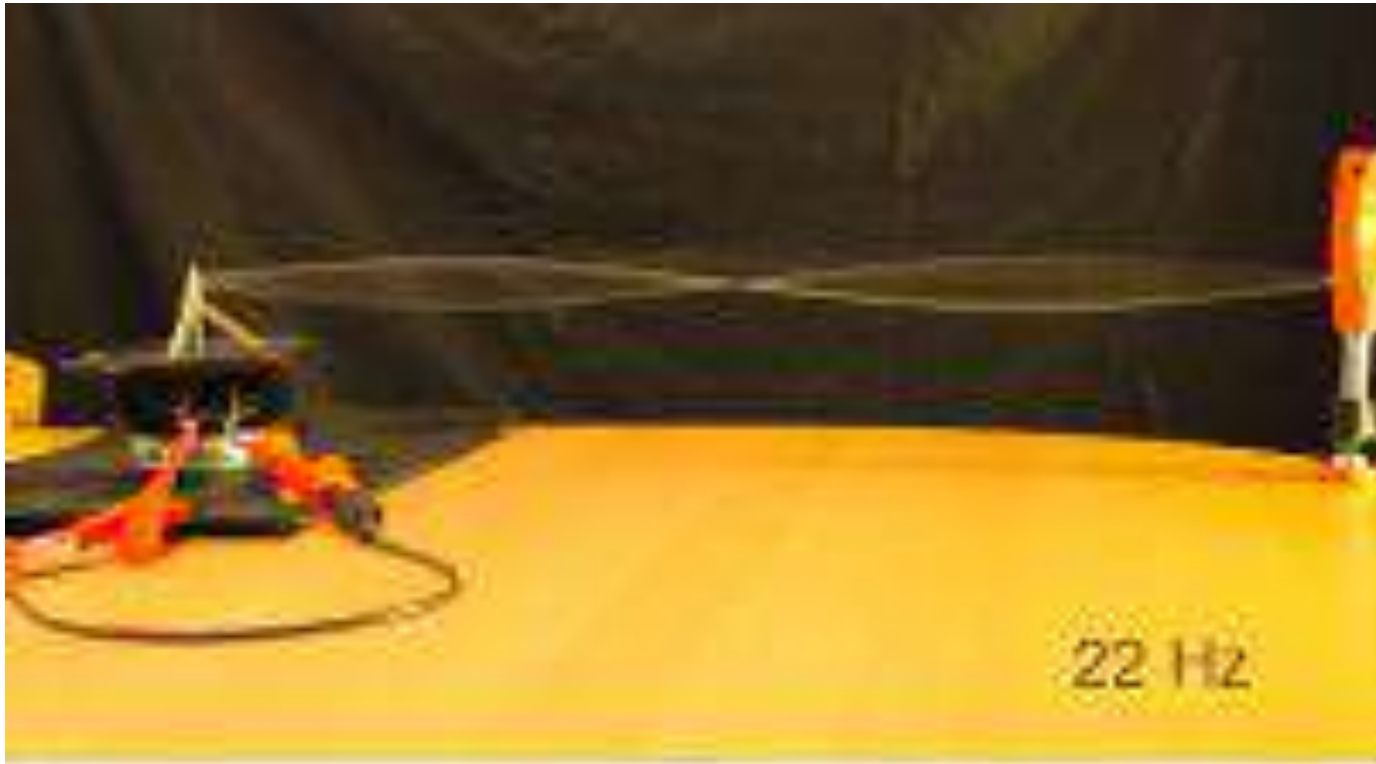
# Initial disturbance



# Initial disturbance



# Periodic forcing



Remark: frequency  
has a great effect



# Periodic forcing



Remark: tension  
has a great effect

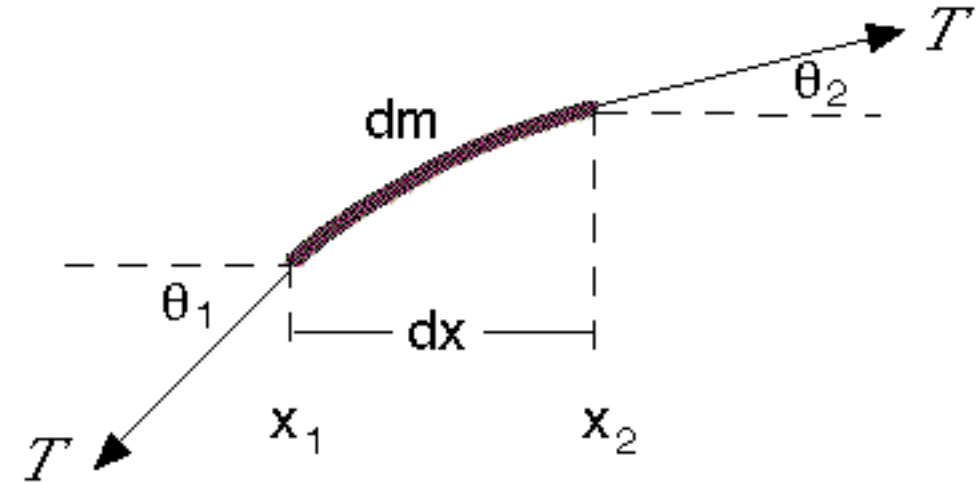
# Basic theory

Problem No. 11 Guitar string

# Basic theory

- $F_y = T \sin(\theta_2) - T \sin(\theta_1)$
- $F_y = T \left( \frac{dy(x_2)}{dx} - \frac{dy(x_1)}{dx} \right) = T \frac{d^2y}{dx^2}$
- $F_y = dm \frac{d^2y}{dt^2} = \mu dx \frac{d^2y}{dt^2}$

$$\boxed{\frac{d^2y}{dt^2} = \frac{T}{\mu} \frac{d^2y}{dx^2}}$$



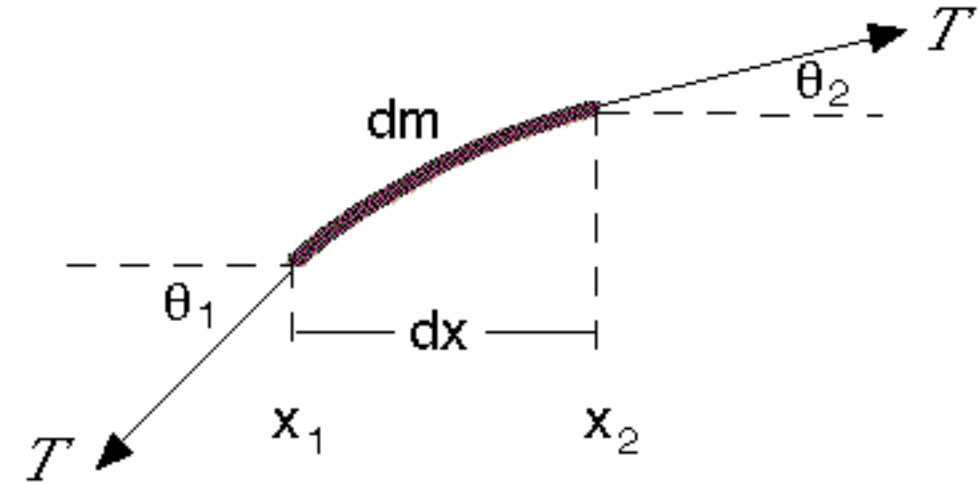
# Basic theory

$$\boxed{\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2} \quad \text{With } \frac{T}{\mu} = c^2}$$

- wave equation, general solution?

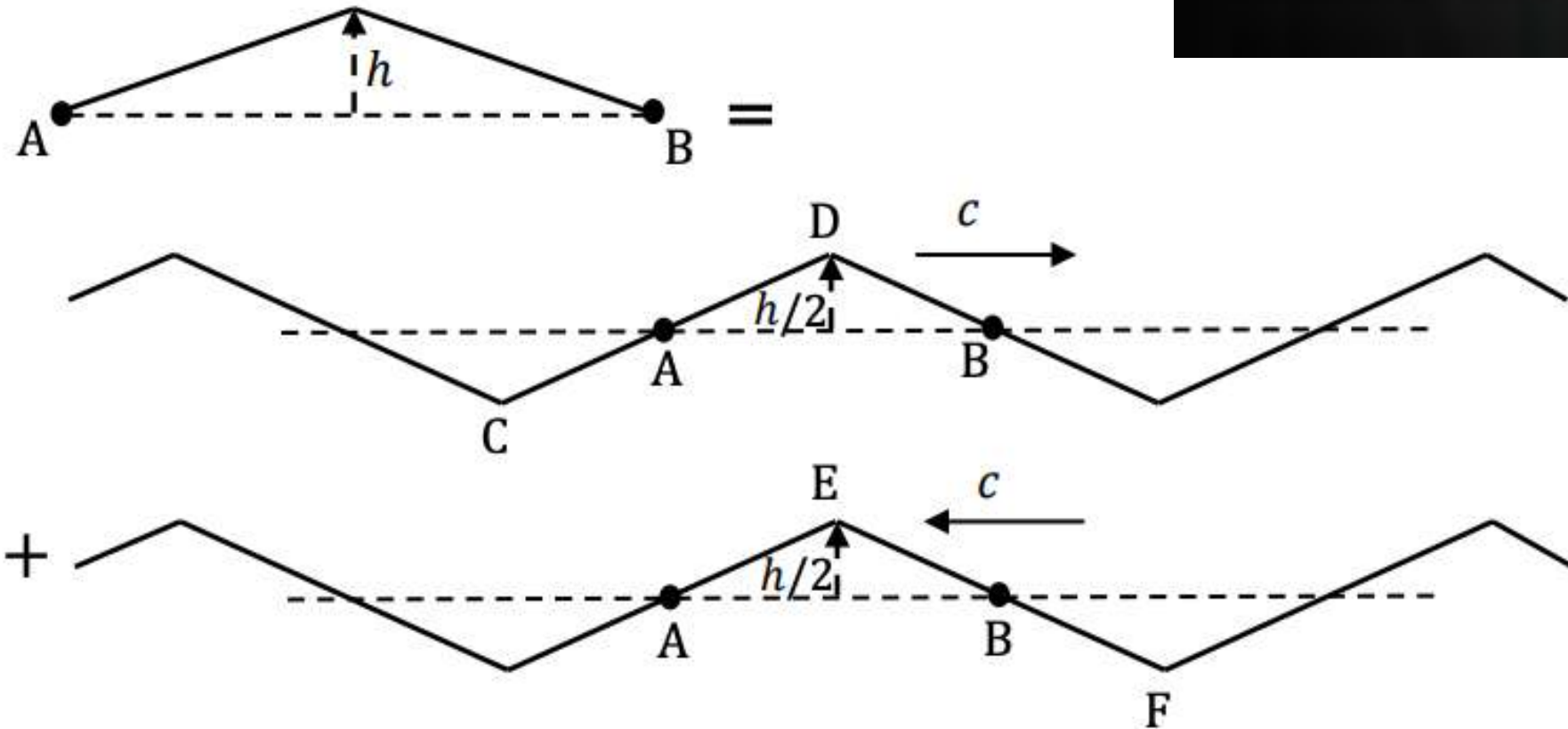
$$\boxed{\cancel{y = A \sin(k(x - ct)) + B \cos(k(x - ct))}} \quad ?$$

$$\boxed{y = F(x - ct) + G(x + ct)}$$



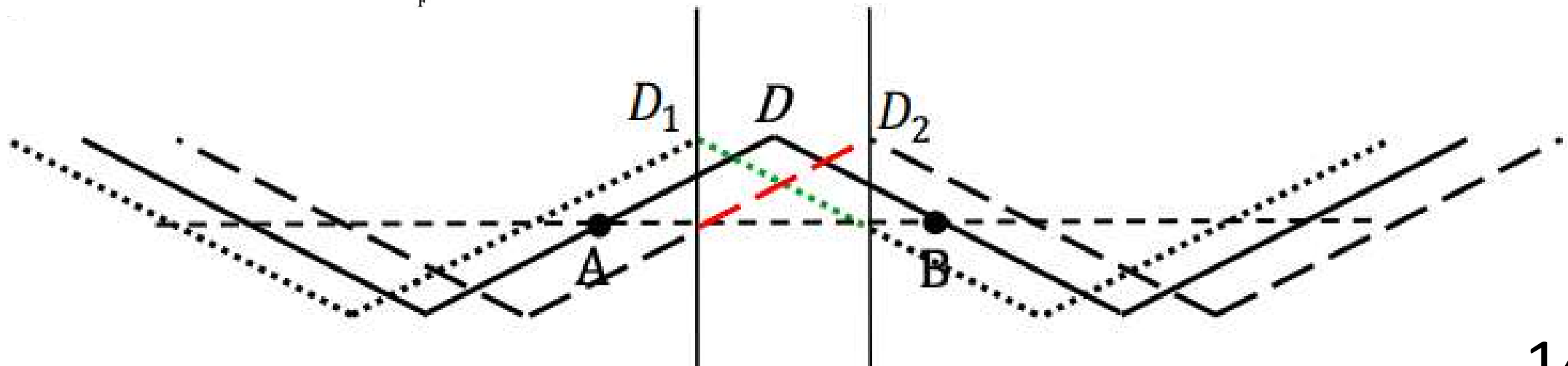
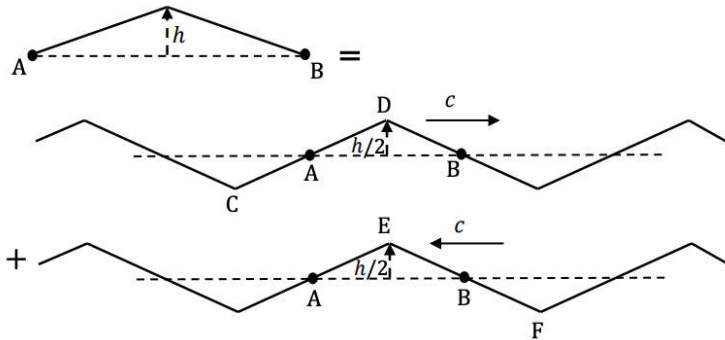
# Basic theory

$$y = F(x - ct) + G(x + ct)$$



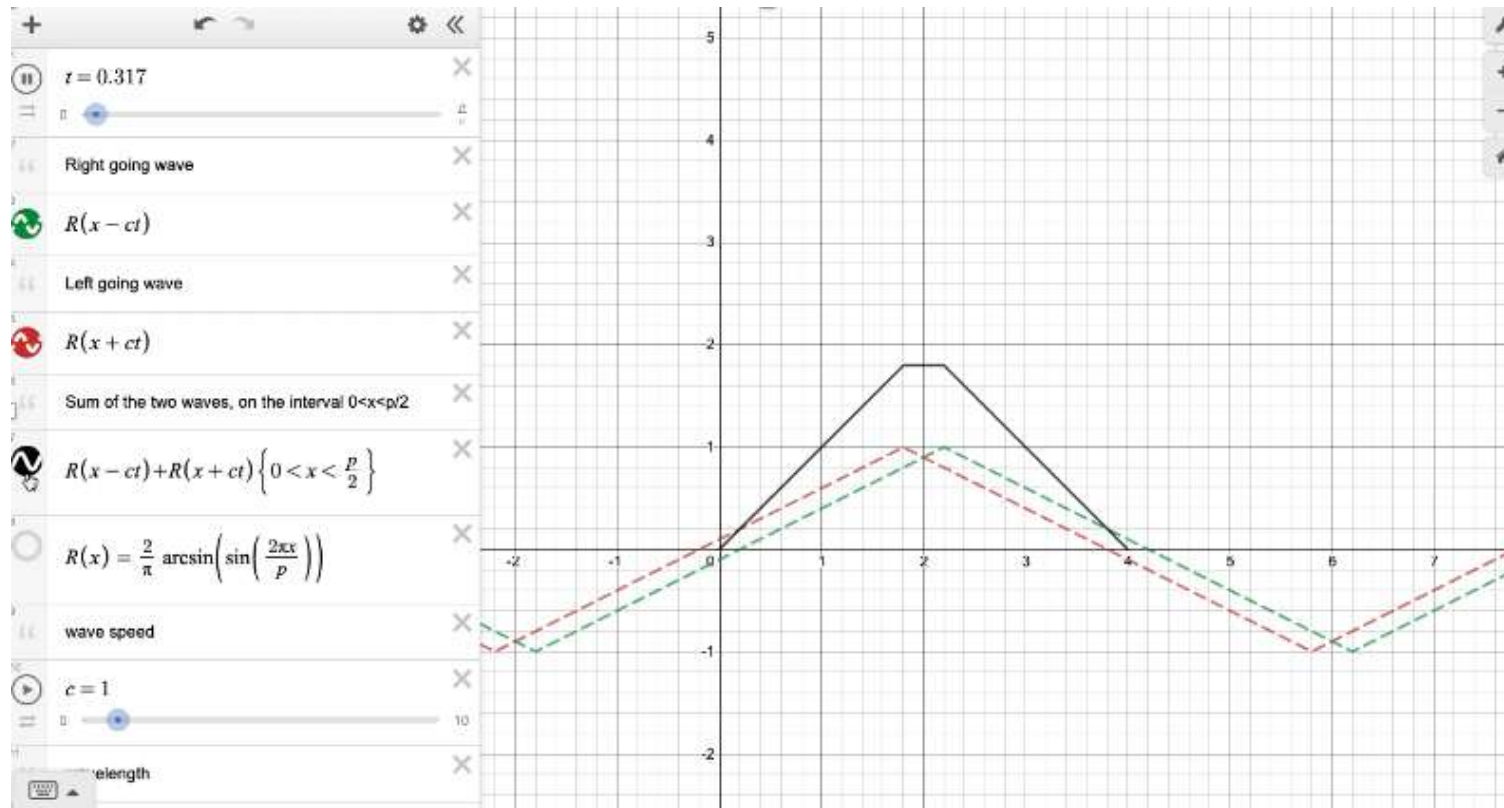
# Basic theory

$$y = F(x - ct) + G(x + ct)$$

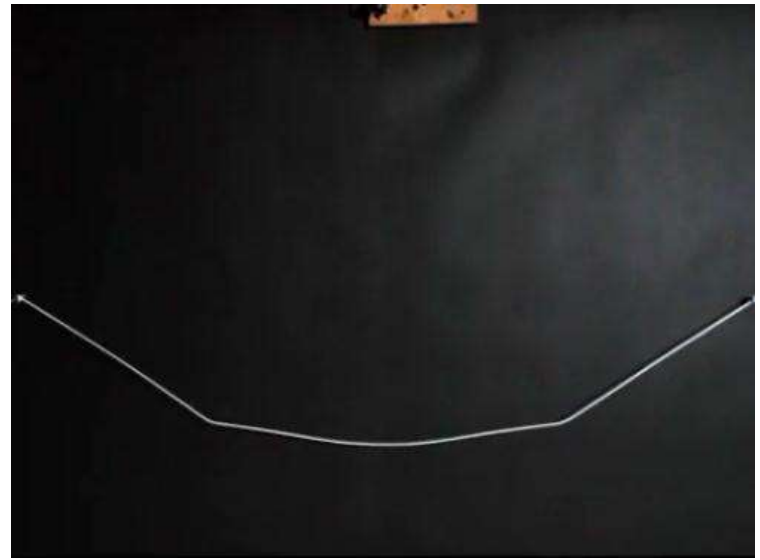


# Basic theory

$$y = F(x - ct) + G(x + ct)$$



<https://www.desmos.com/calculator/365anbv6mq>



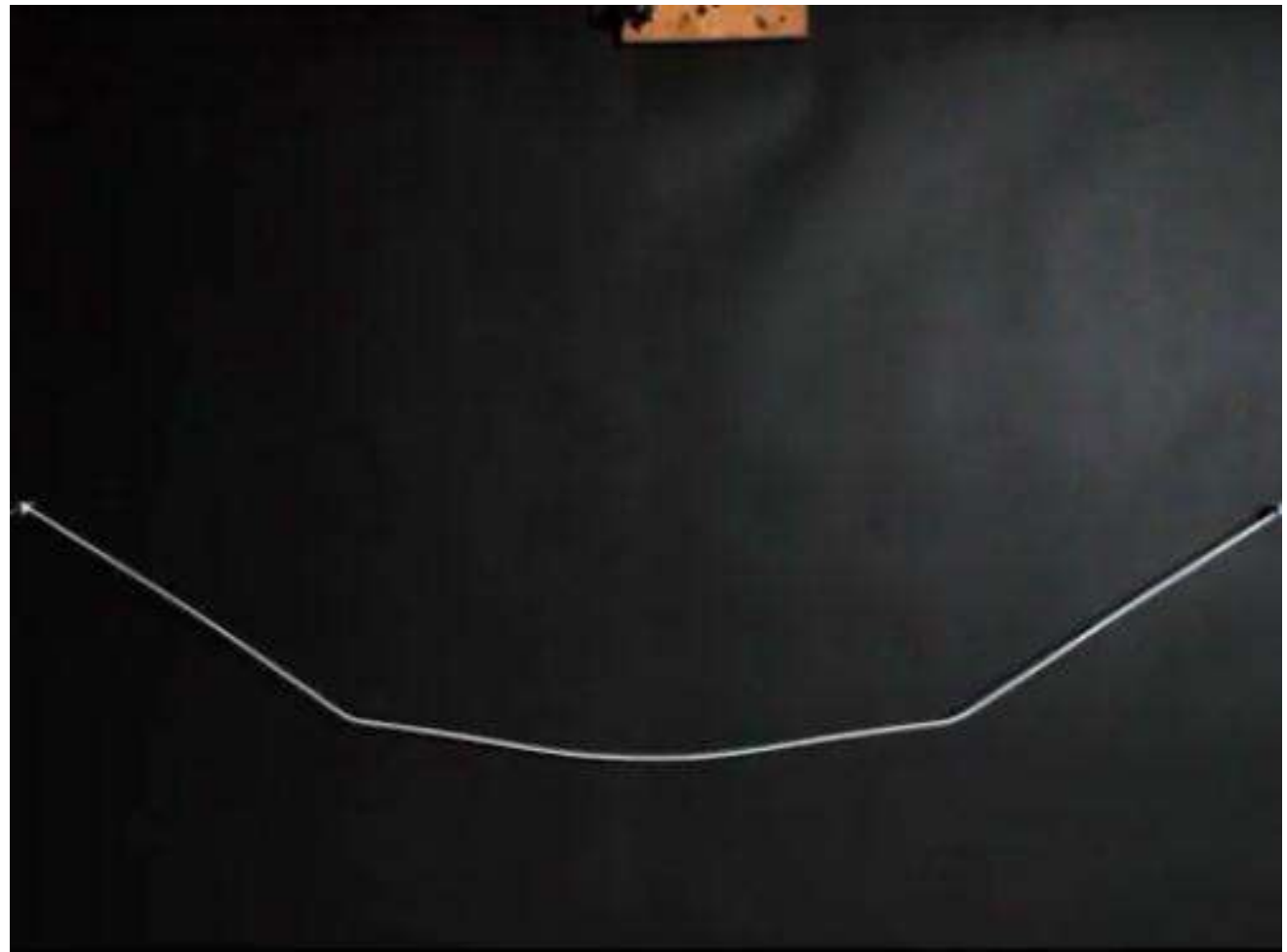
# Basic Theory Limitations

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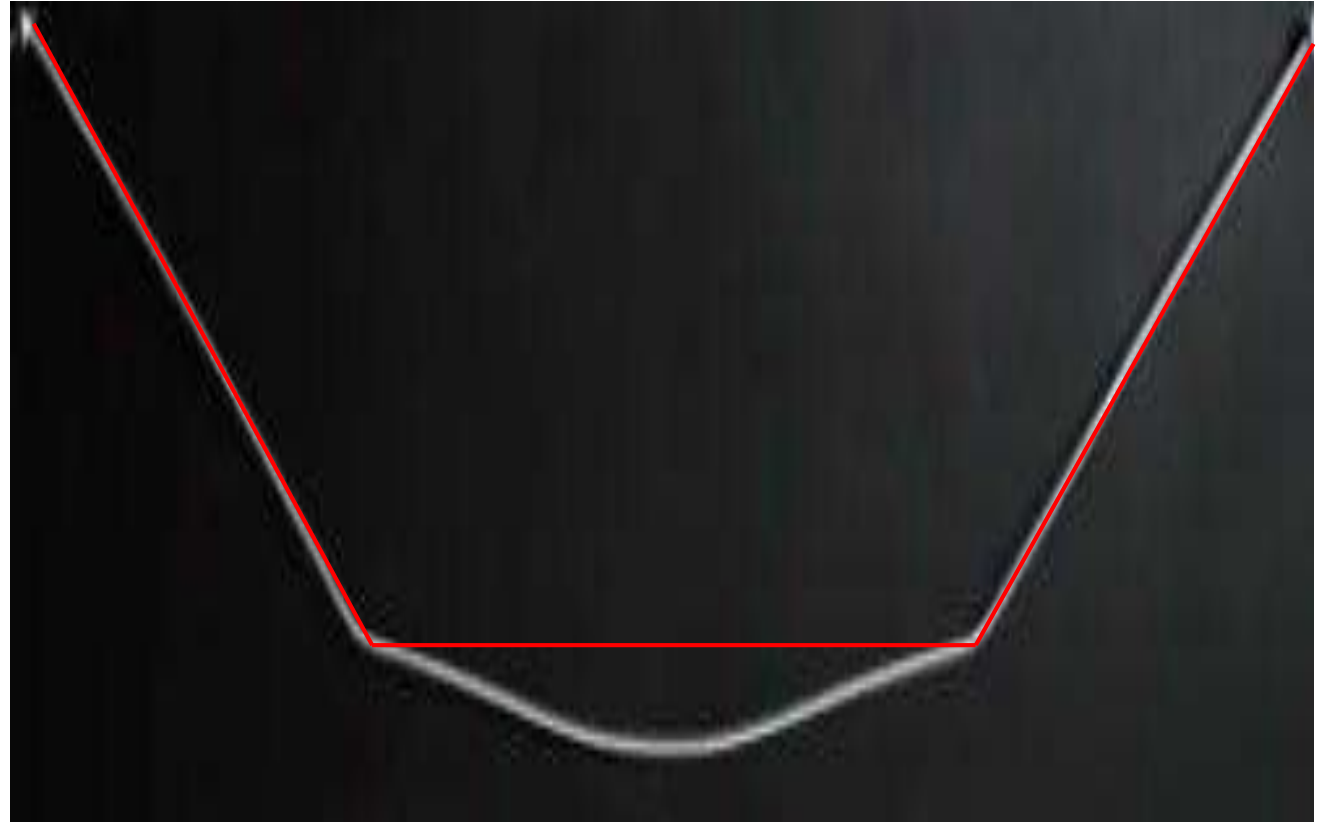
# Limitation 1

$$y = F(x - ct) + G(x + ct)$$



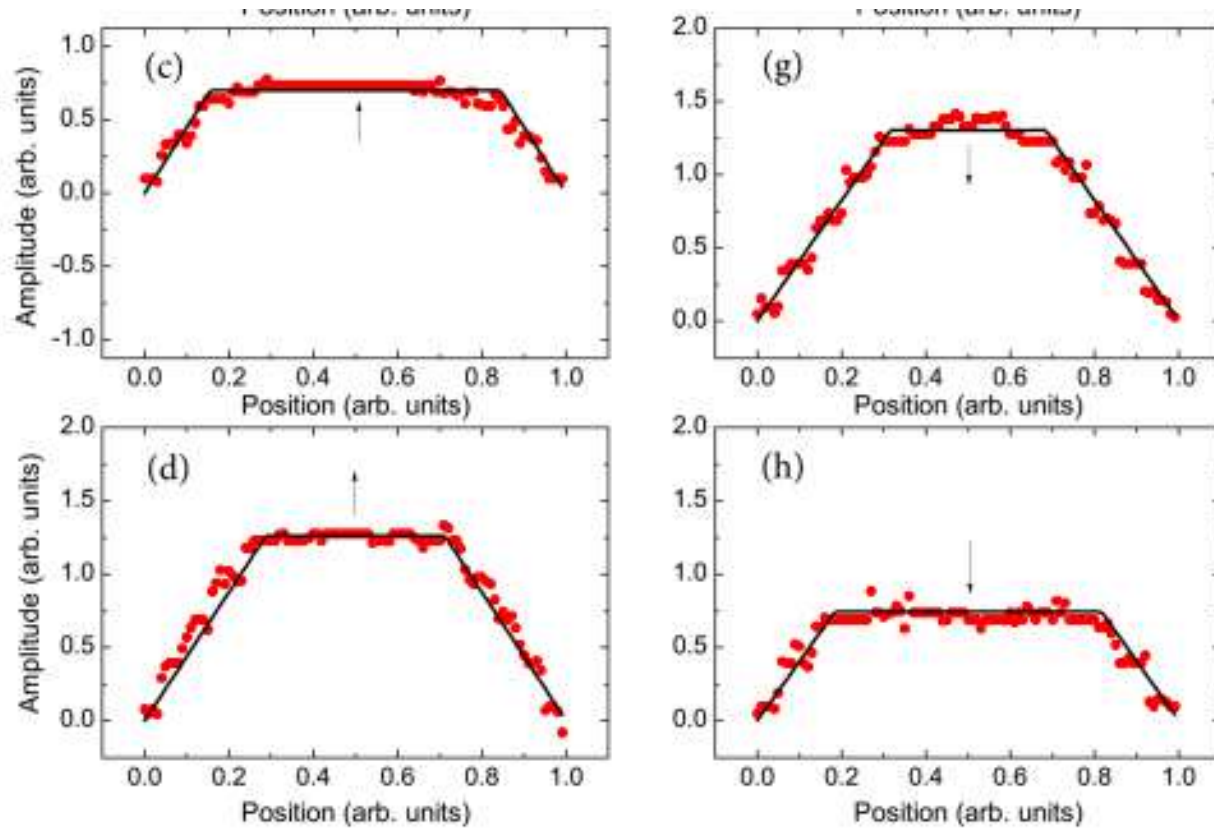
# Limitation 1

$$y = F(x - ct) + G(x + ct)$$

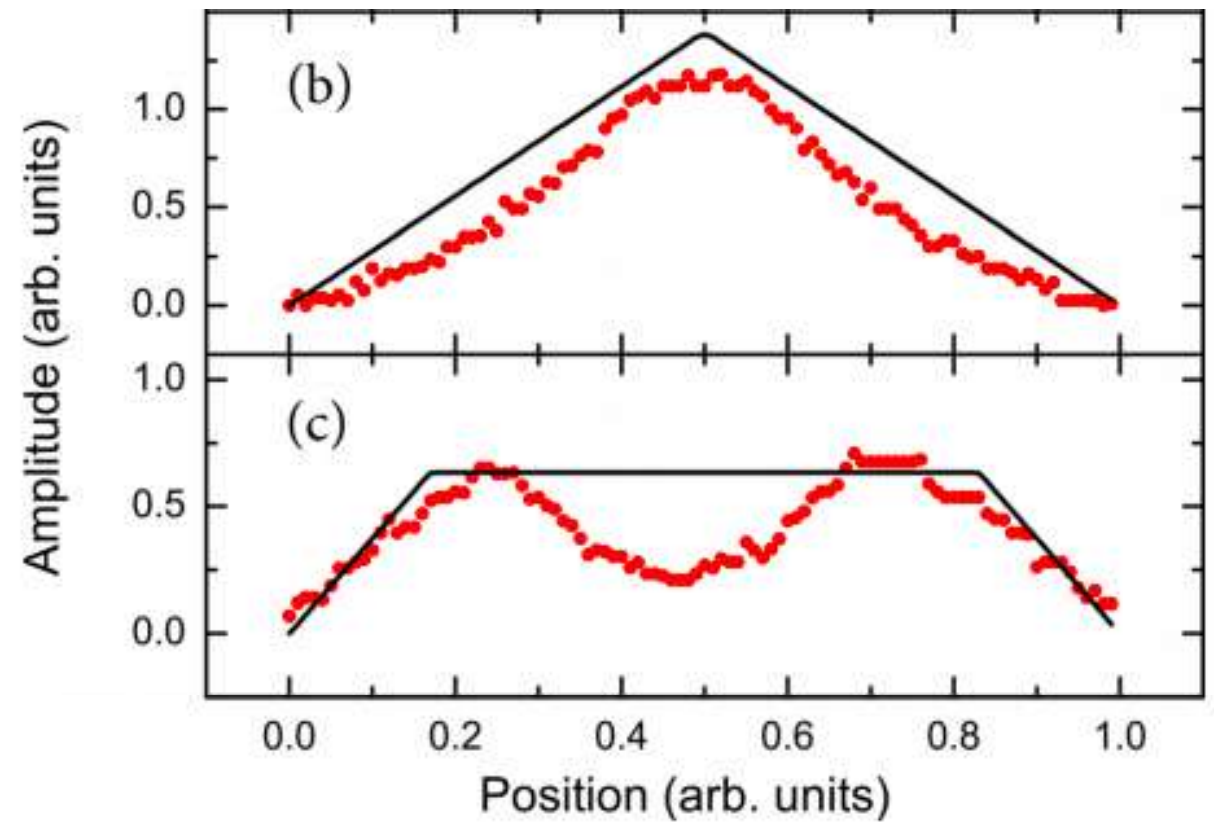


# Limitation 1

## Small amplitudes



## Big amplitudes

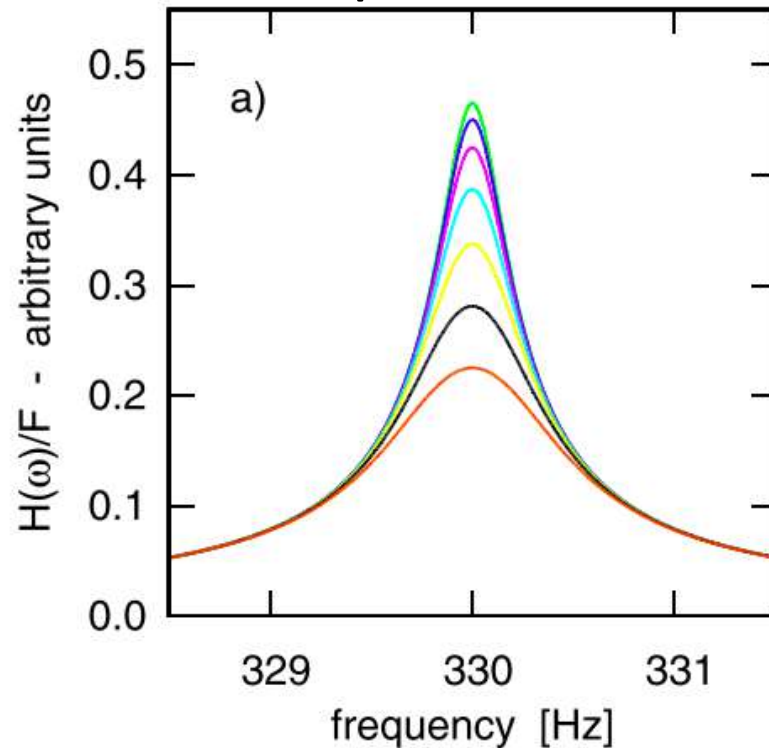


# Limitation 2

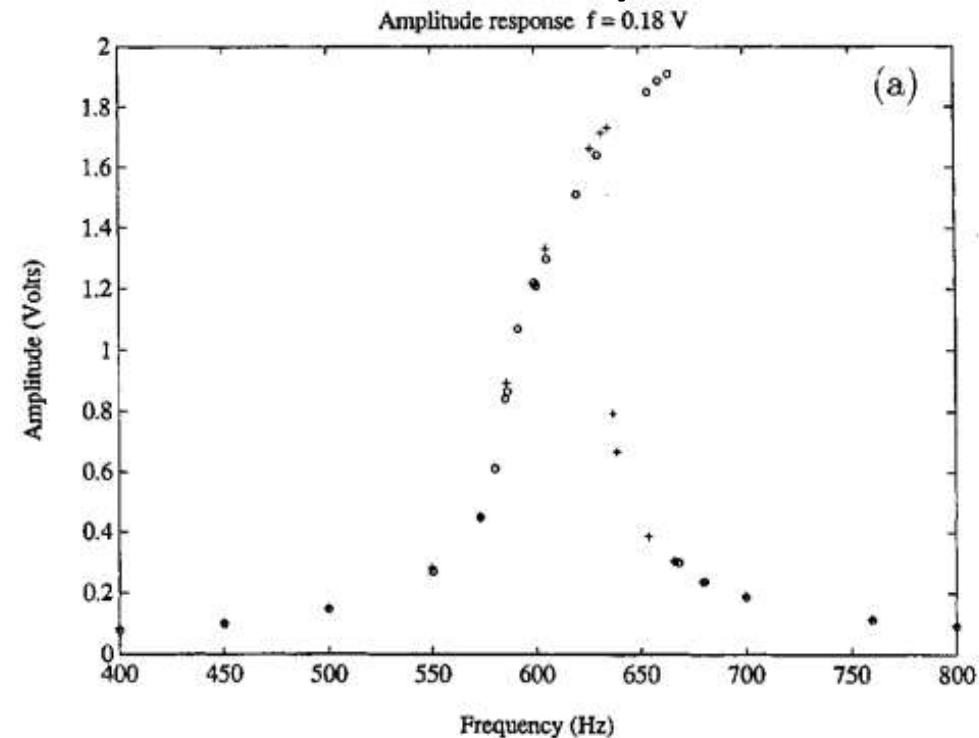
$\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2} + F(x, t)$	With $\frac{T}{\mu} = c^2$
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- Considering an external periodic forcing  $F(x, t)$

Expectation



Reality



# Basic theory

What went wrong?

...

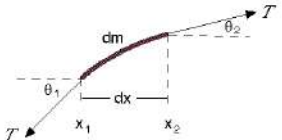
Basic theory

- $F_y = T \sin(\theta_2) - T \sin(\theta_1)$
- $F_y = T \left( \frac{dy(x_2)}{dx} - \frac{dy(x_1)}{dx} \right) = T \frac{d^2 y}{dx^2}$
- $F_y = dm \frac{d^2 y}{dt^2} = \mu dx \frac{d^2 y}{dt^2}$

$$\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2}$$

11

...



We assumed some things

- Constant tension
- Small amplitudes
- Planar motion

# Past literature review

Problem No. 11 Guitar string

# Oplinger 1960

“Modified wave equation for transverse motion”:

$$\frac{d^2 y}{dt^2} = \frac{T_0}{\mu} \frac{d^2 y}{dx^2} + \frac{1}{2} \frac{EA}{\mu} \left[ \frac{1}{L} \int_0^L \left( \frac{dy}{dx} \right)^2 dx \right] \frac{d^2 y}{dx^2}$$

$$\frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2}$$

Accounts for tension changes

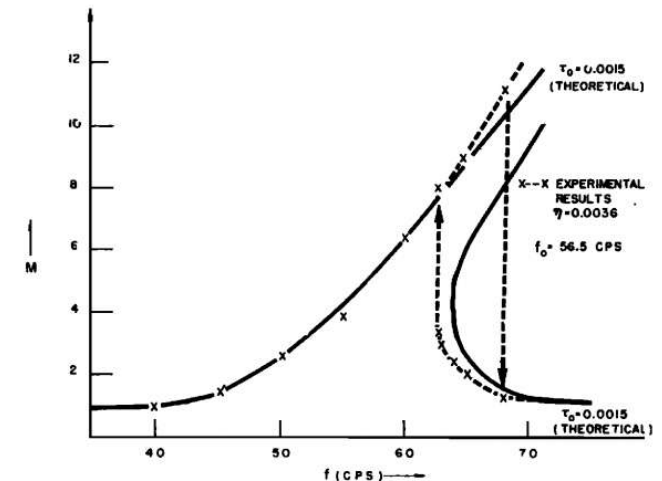


FIG. 10. Comparison of theoretical and experimental results for a typical case. Note the presence of observed jump frequencies.

# Oplinger 1960

“Modified wave equation for transverse motion”:

$$\frac{d^2 y}{dt^2} = \frac{T_0}{\mu} \frac{d^2 y}{dx^2} + \frac{1}{2} \frac{EA}{\mu} \left[ \frac{1}{L} \int_0^L \left( \frac{dy}{dx} \right)^2 dx \right] \frac{d^2 y}{dx^2}$$

Accounts for tension changes

Only one transverse direction

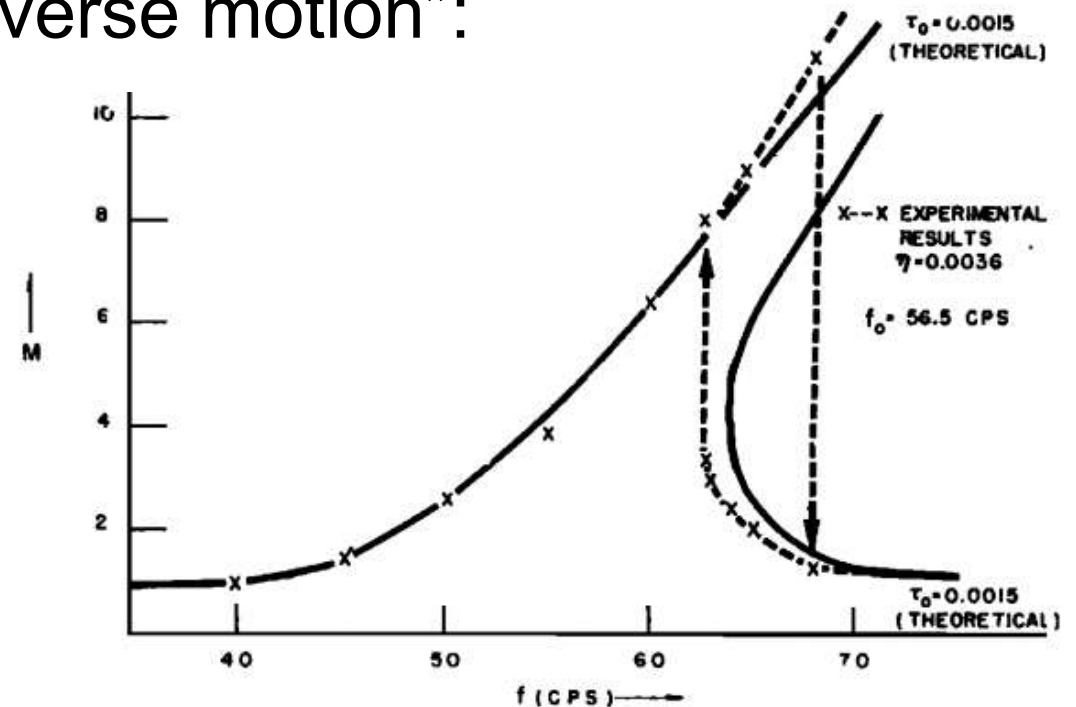


FIG. 10. Comparison of theoretical and experimental results for a typical case. Note the presence of observed jump frequencies.



# Oplinger 1960

- "For these and other curves it was found convenient to write a program for a Bendix G-15 computer"

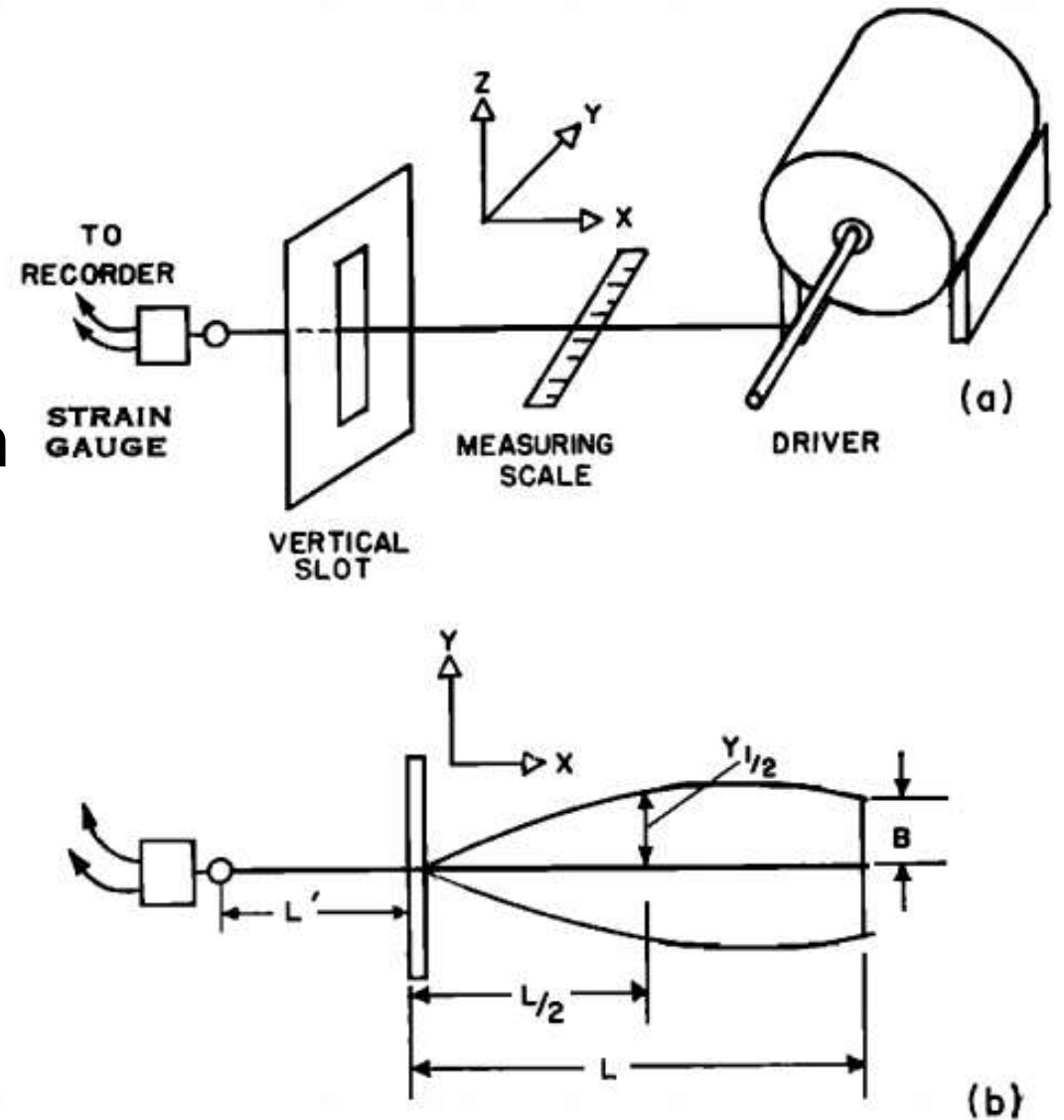


FIG. 9. (a) Experimental apparatus; (b) basic parameters.

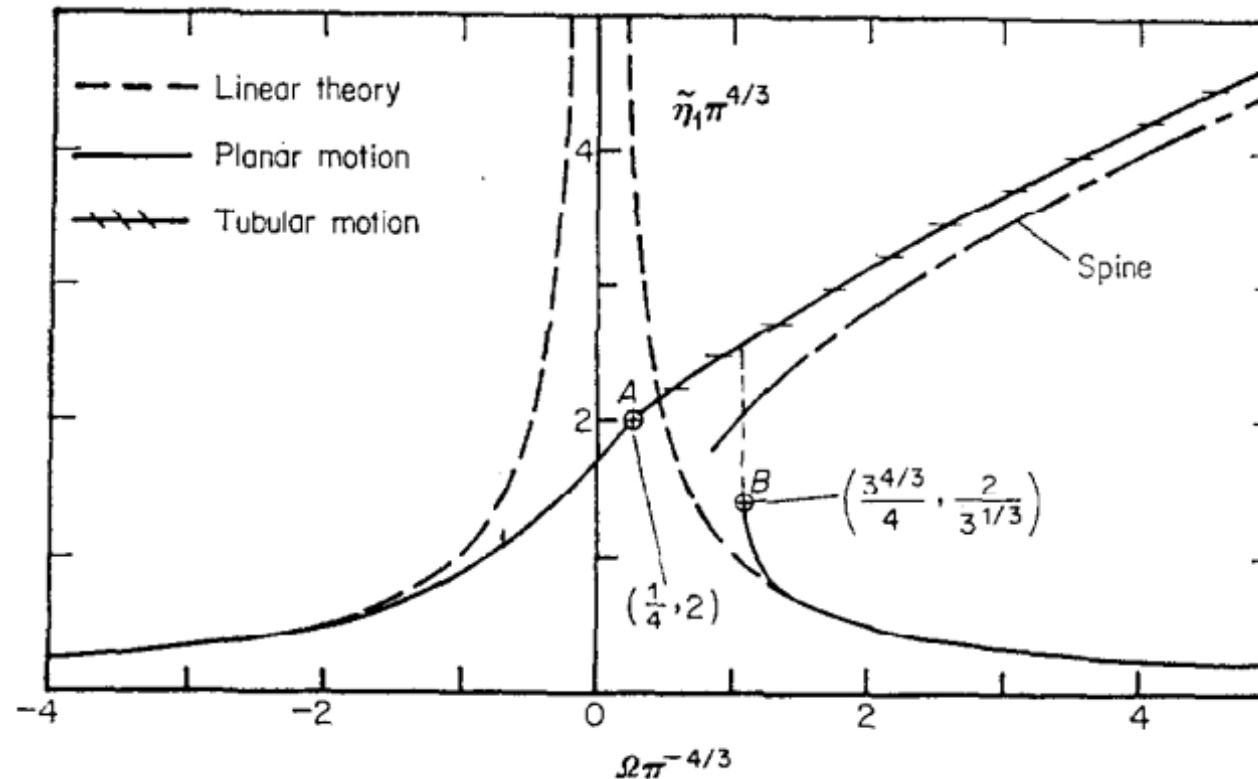
# Narasimha 1968

- Found the exact equations for 3D motion.

$$(1 - u_x) \ddot{u} - c_1^2 \frac{\partial \Lambda}{\partial x} + h,$$

$$(1 - u_x) \ddot{v} - c_1^2 \frac{\partial}{\partial x} (v_x \Lambda) + \mathbf{f} + \mathbf{g},$$

$$\Lambda \equiv \frac{1 + c_1^2 \lambda - c_2^2 \lambda^2 + c_3^2 \lambda^3}{c_1^2 (1 + \lambda) (1 - u_x)},$$



# Narasimha 1968

- Non-planar oscillations are parametrically excited by planar oscillations (found Mathieu eq.)

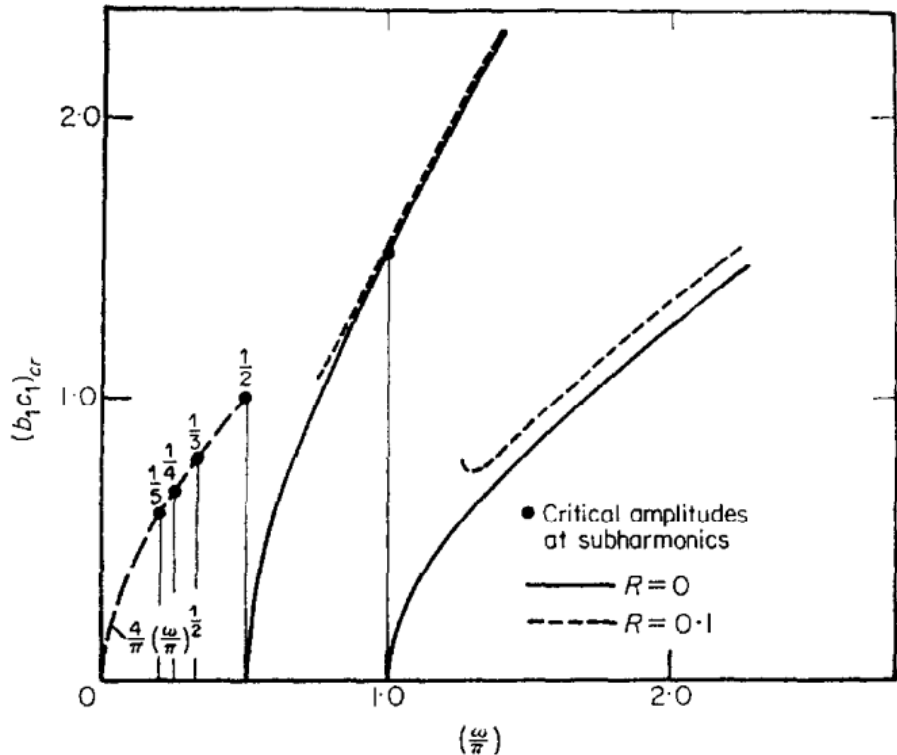
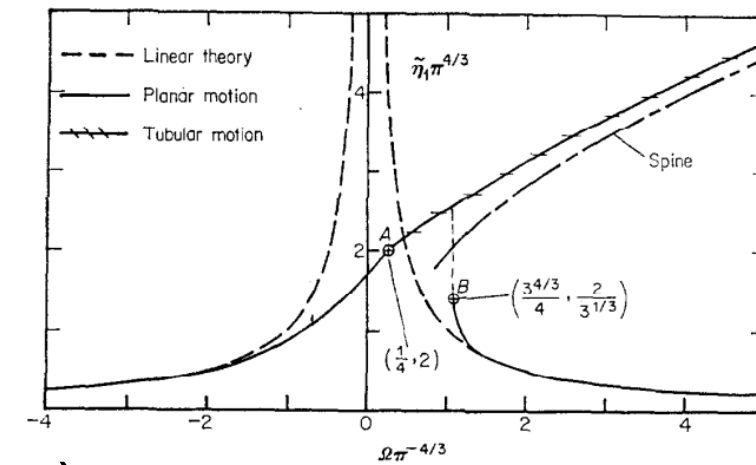


Figure 4. Critical amplitudes for onset of a second mode of transverse motion. With no damping ( $R = 0$ ), the critical amplitude generally increases with frequency, but drops suddenly to zero at the natural frequency and all subharmonics. The largest critical amplitudes necessary for instability occur at frequencies just below these, and are shown by full circles, with numbers to indicate the subharmonic. The dashed line through the origin is the locus of these peak amplitudes and has the asymptotic form  $(b_1 c_1)_{cr} \approx 4\omega^{1/2} \pi^{-3/2}$  as  $\omega \rightarrow 0$ .



# Elliot 1980

Go general, then approximate down.

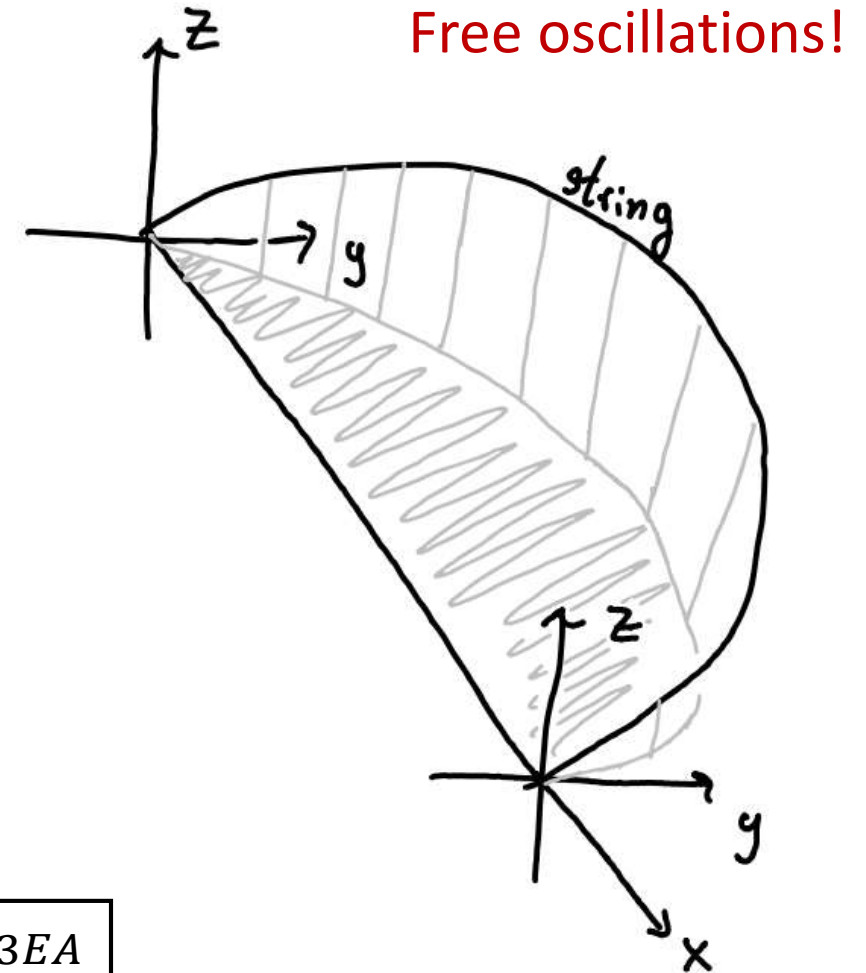
Find energy in terms of the string shape:

$$\begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} = \sum_j \begin{pmatrix} V_{xj} \\ V_{yj} \end{pmatrix} \sin\left(\frac{j\pi x}{L}\right)$$

now plug into energy equation and obtain:

$\begin{aligned} \ddot{Y} + \omega^2 Y [1 + \sigma(Y^2 + Z^2)] &= 0 \\ \ddot{Z} + \omega^2 Z [1 + \sigma(Y^2 + Z^2)] &= 0 \end{aligned}$	$\omega = \frac{T_0}{2\mu L}, \quad \sigma = \frac{3EA}{8T_0}$
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$F$  represents the first Fourier component of force



# Elliot 1980

Taking limit of  $Z \ll Y$

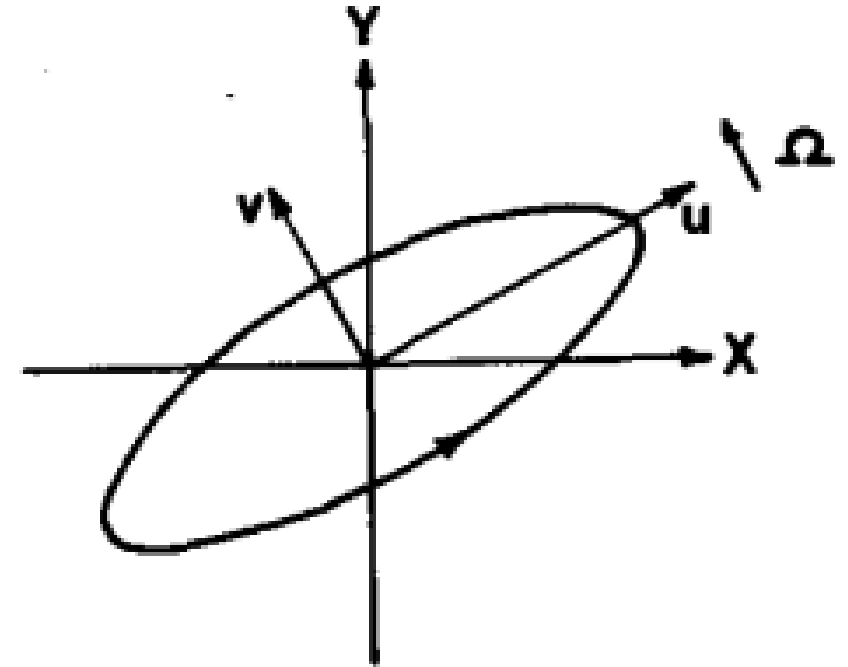
$$\begin{aligned}\ddot{Y} + \omega^2 Y[1 + \sigma Y^2] &= 0 \\ \ddot{Z} + \omega^2 Z[1 + \sigma Y^2] &= 0\end{aligned}$$

Got Duffing eq. in  $Y$  solved (approx.) by:

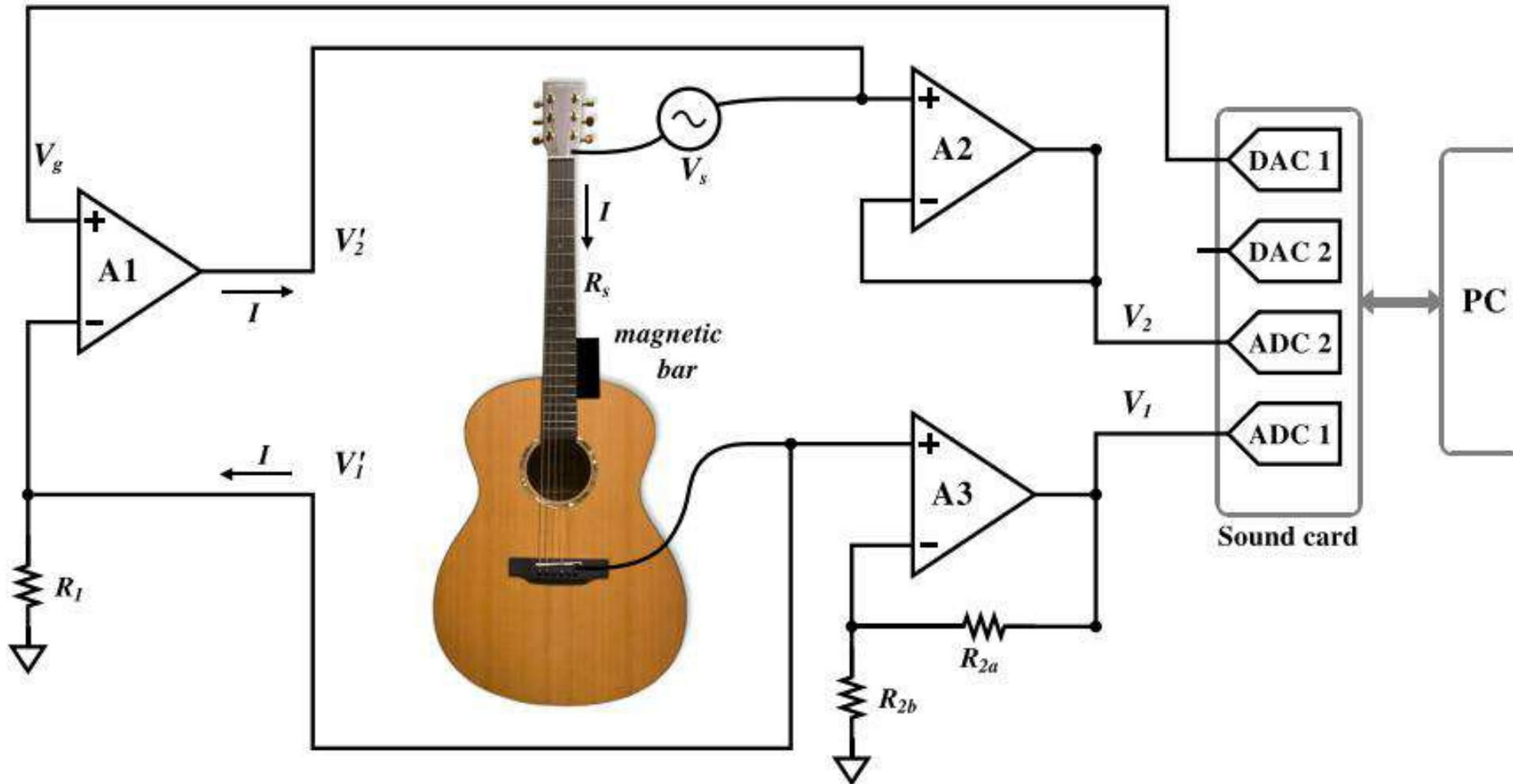
$$y = a \cos(pt) \quad p = \omega(1 + \sigma a^2/2)$$

Then in the other direction have:

$$\ddot{Z} + \omega^2 Z \left[ 1 + \frac{\sigma a^2}{2} (1 + \cos(2pt)) \right] = 0$$



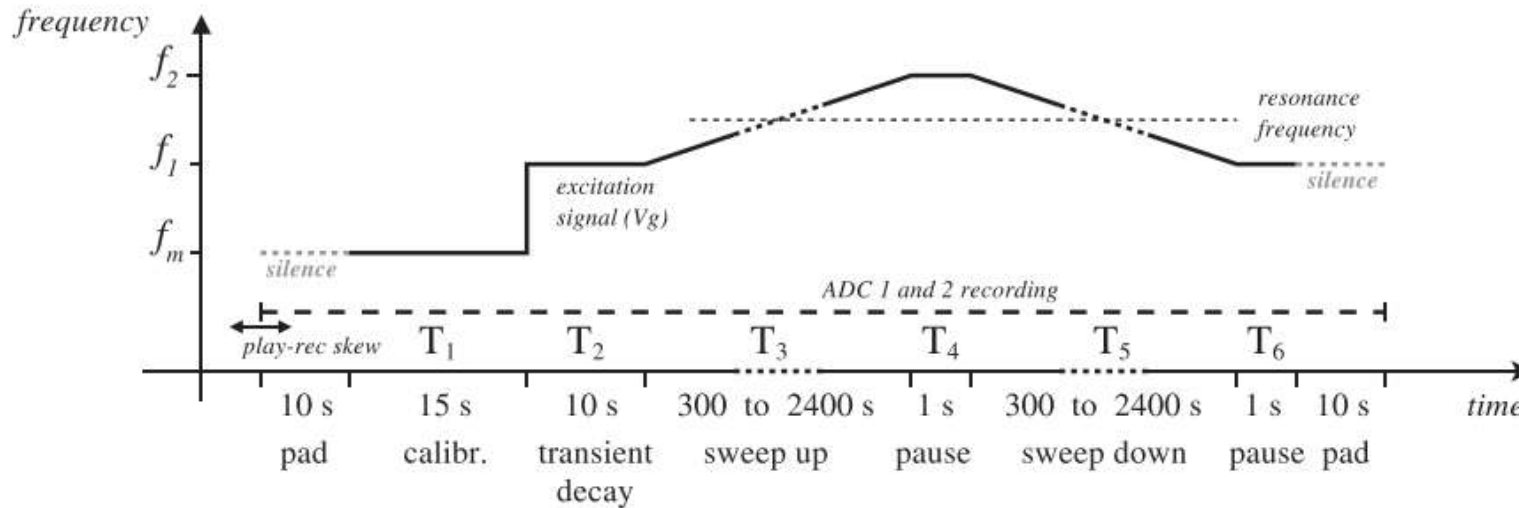
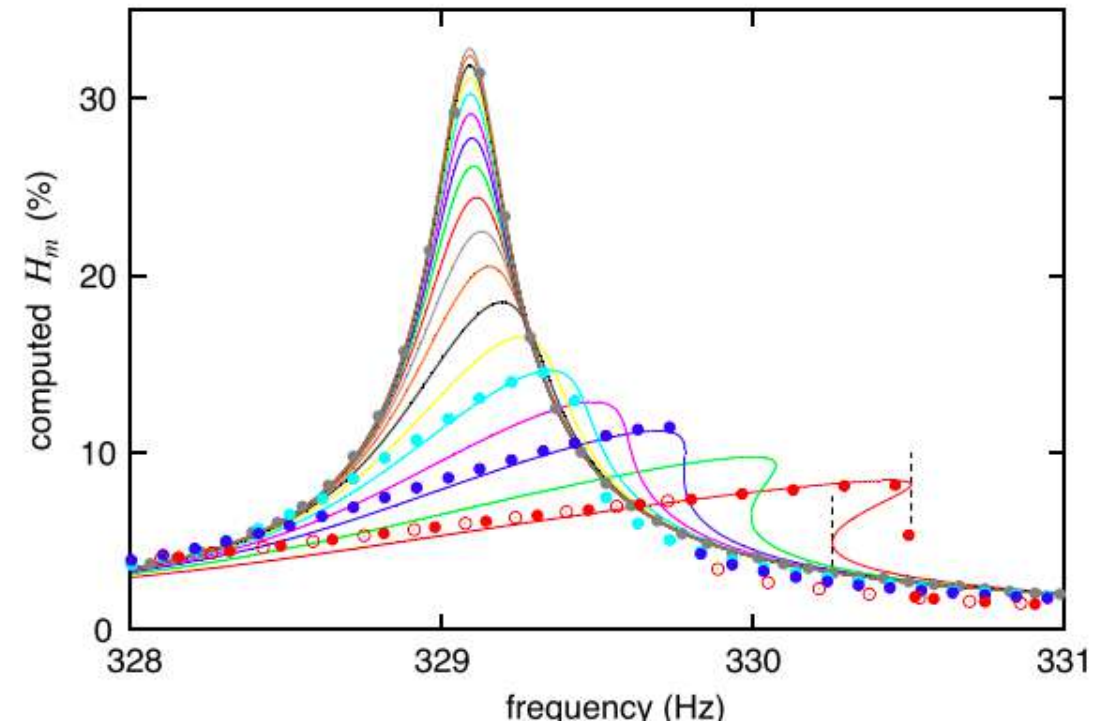
# Carla 2017



# Carla 2017

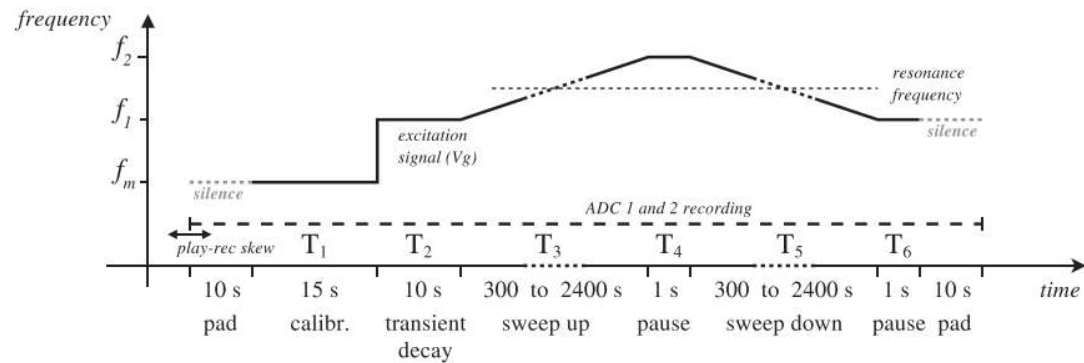
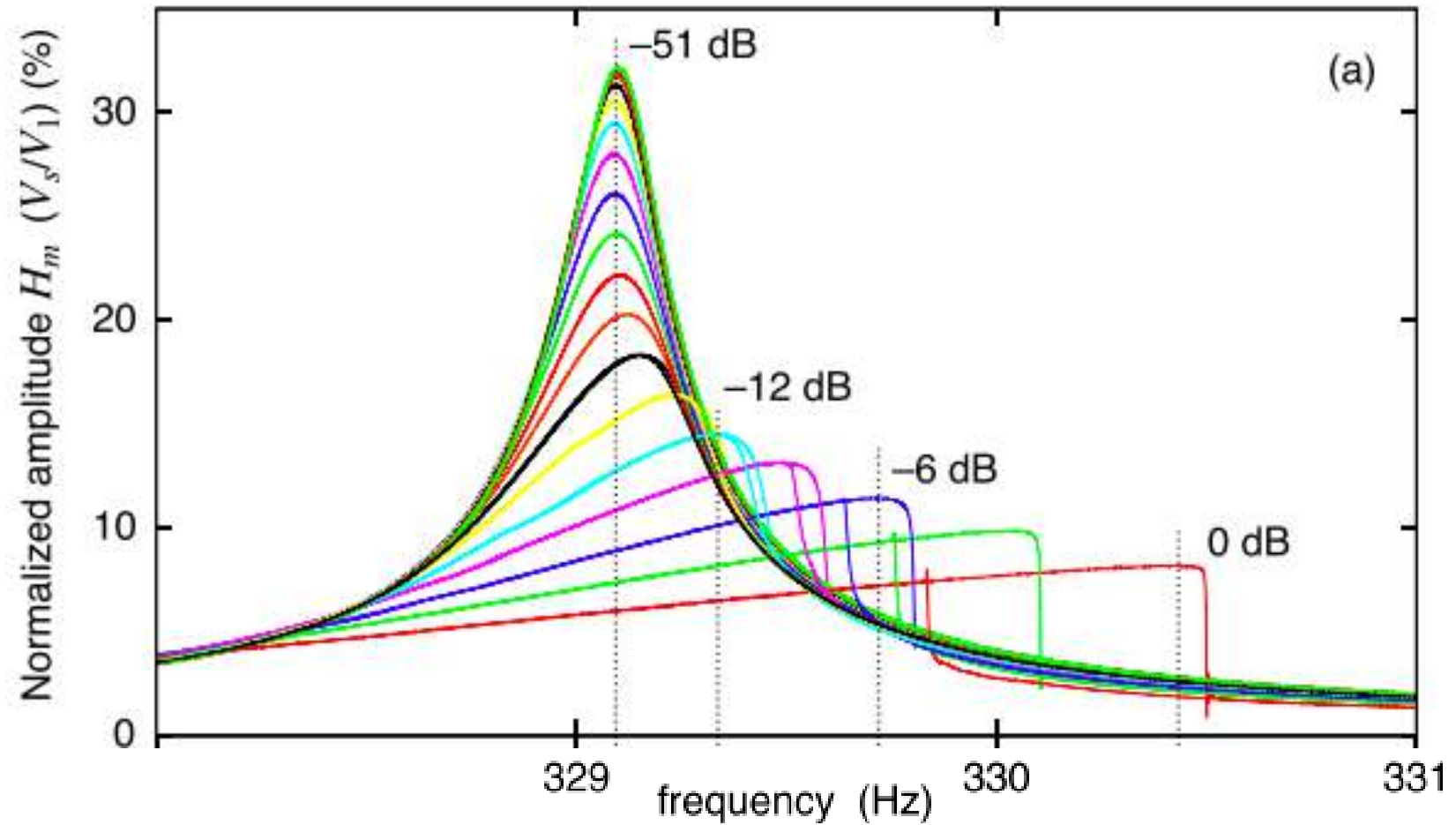
$$\ddot{Y} + \omega\beta\dot{Y} + \omega^2Y[1 + \sigma Y^2] = 0$$

Comparing experiments with the  
damped Duffing equation



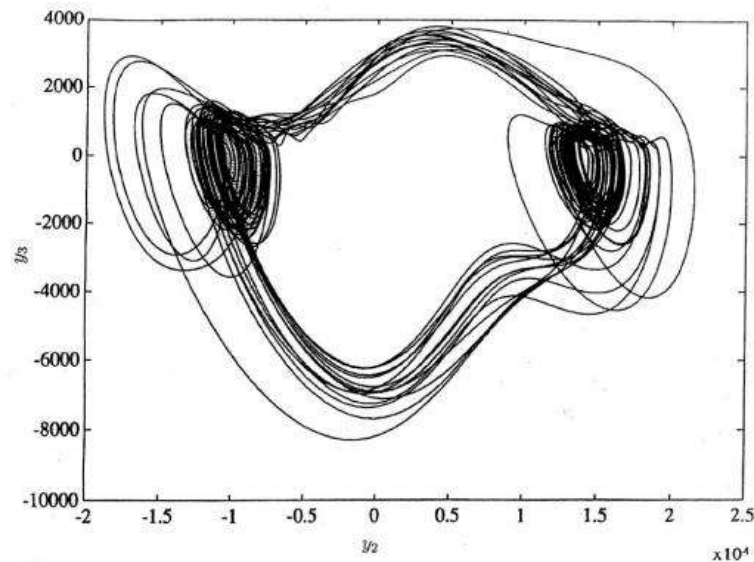
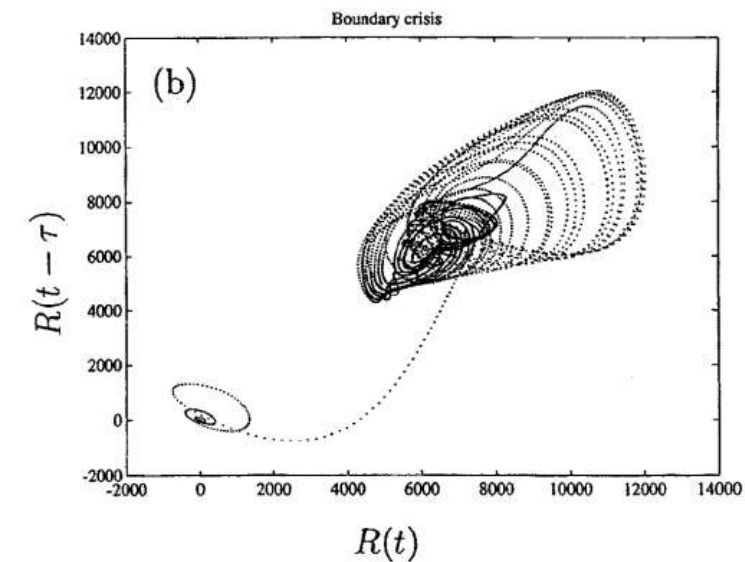
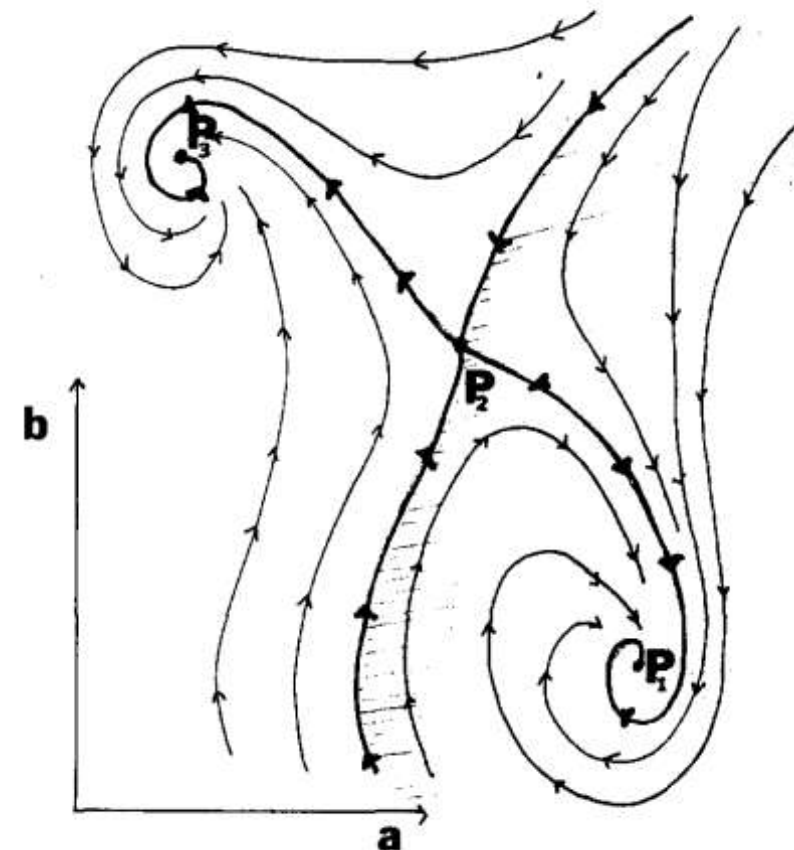
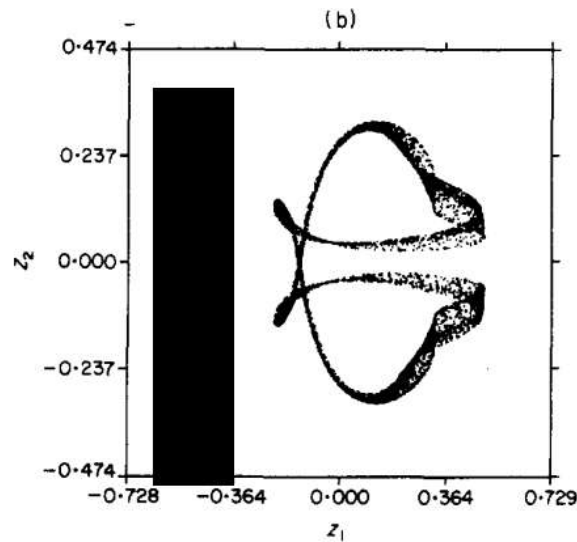
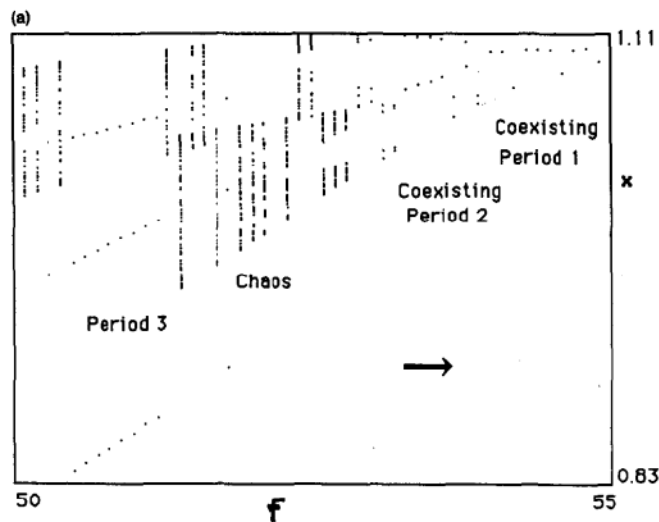


# Carla 2017





# And a lot of chaos theory...



# Setup

Problem No. 11 Guitar string

# Setup - forcing

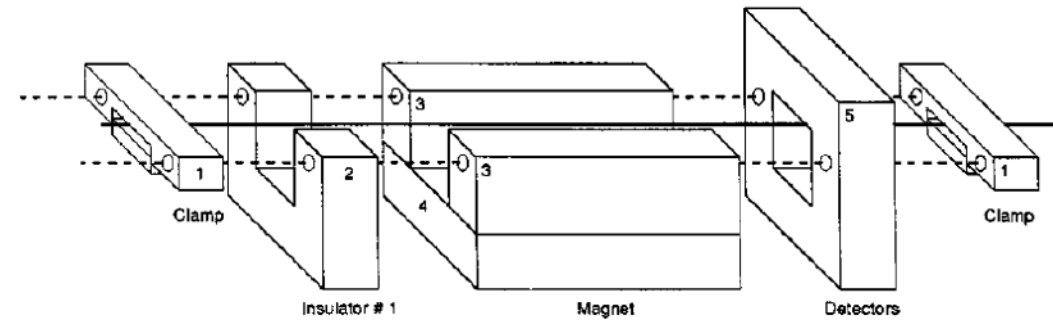


Fig. 1. Schematic of the string mounting system.

- Most of modern literature uses alternating current in string + a magnet to force the string
  - How does this fit the pr. statement?
  - Use non-ferromagnetic wire for non-planar motions
- Could also use ferromagnetic string and oscillating magnetic field from an electromagnet
- A mechanical forcing should reproduce the effects (see ref.), but violates pr. statement (?)

# Setup – detection 1

- Use stroboscope to “freeze vibrations”
  - Then capture with a camera

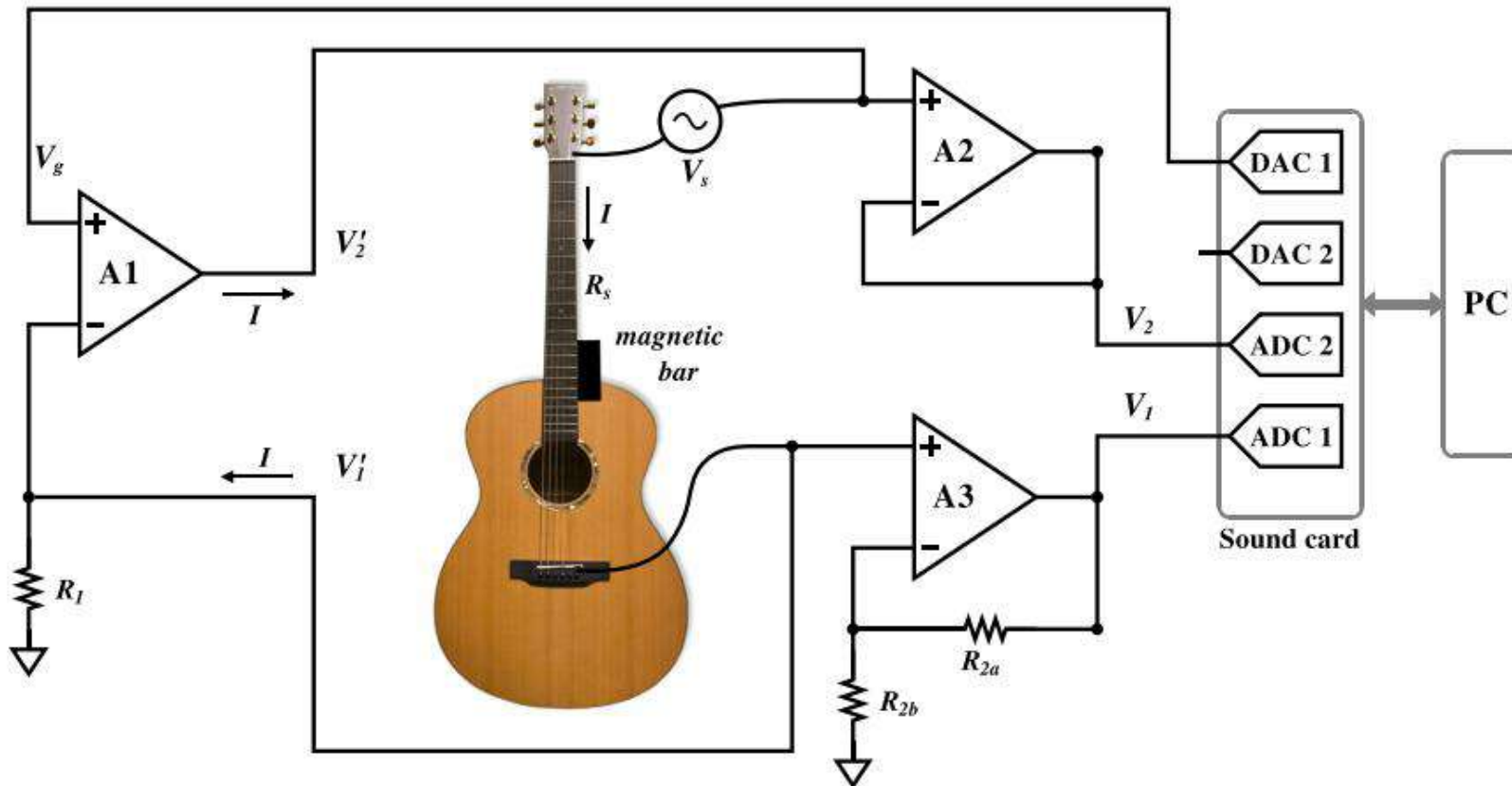


# Setup – detection 2

- Either use very high-speed camera (slow motion),
- or capture a large amount of photos with high shutter speed
  - then pick one with highest amplitude,
- or take photos with slow enough shutter speeds
  - $\sim$  shutter speed (time)  $\gtrsim 1/\text{frequency}$  of vibrations to capture a whole period
  - motion-blurred, but maxima visible

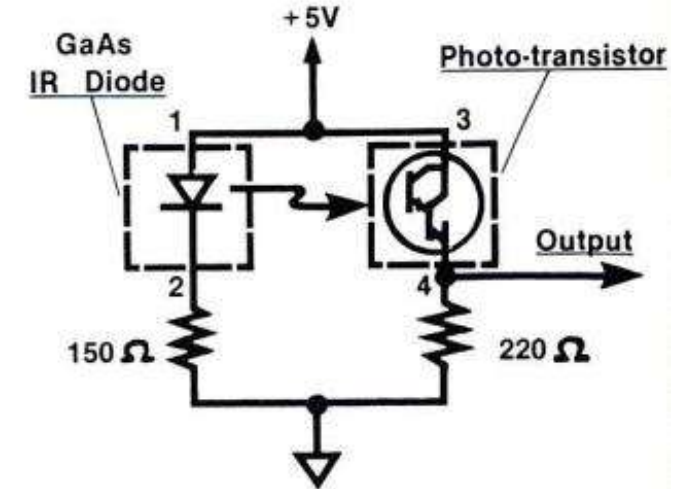
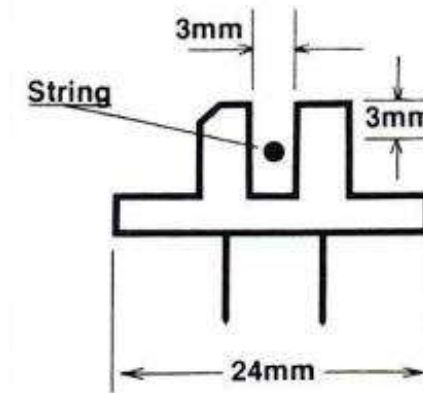
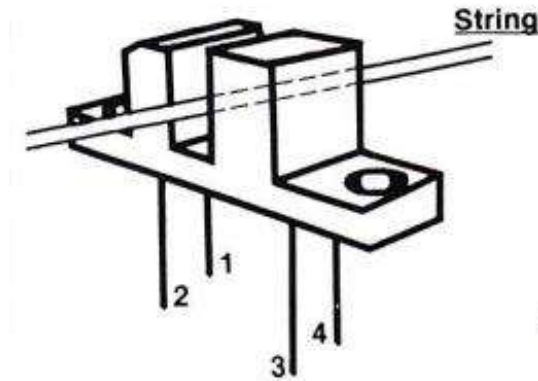
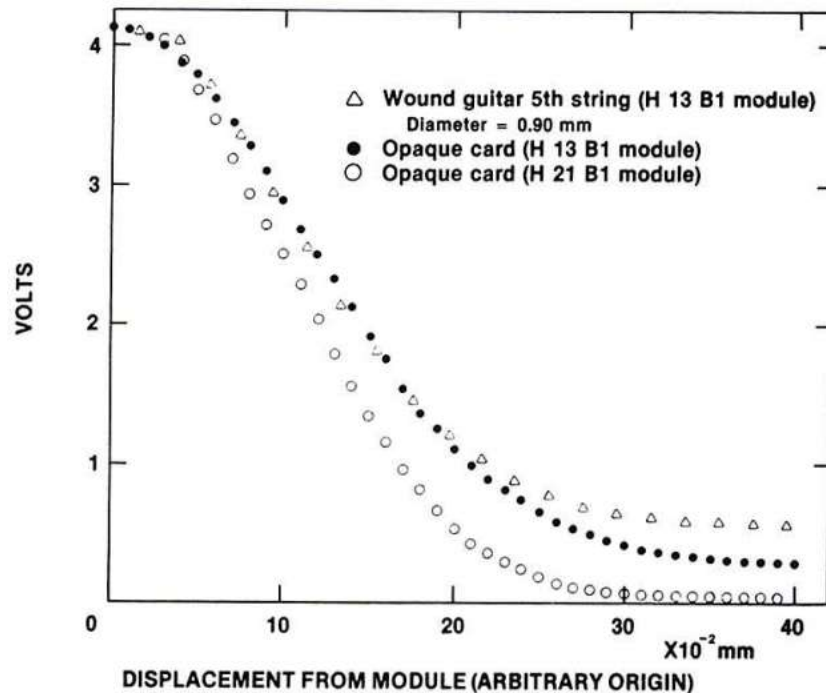
# Setup – detection 3

- Accurate, but only measures amplitude in 1 t. direction



# Setup – detection 4

- can have 2 of these,
  - measure amplitude in both transverse



# Conclusion

What you should work on

Problem No. 11 Guitar string



# To measure: Duffing response curve

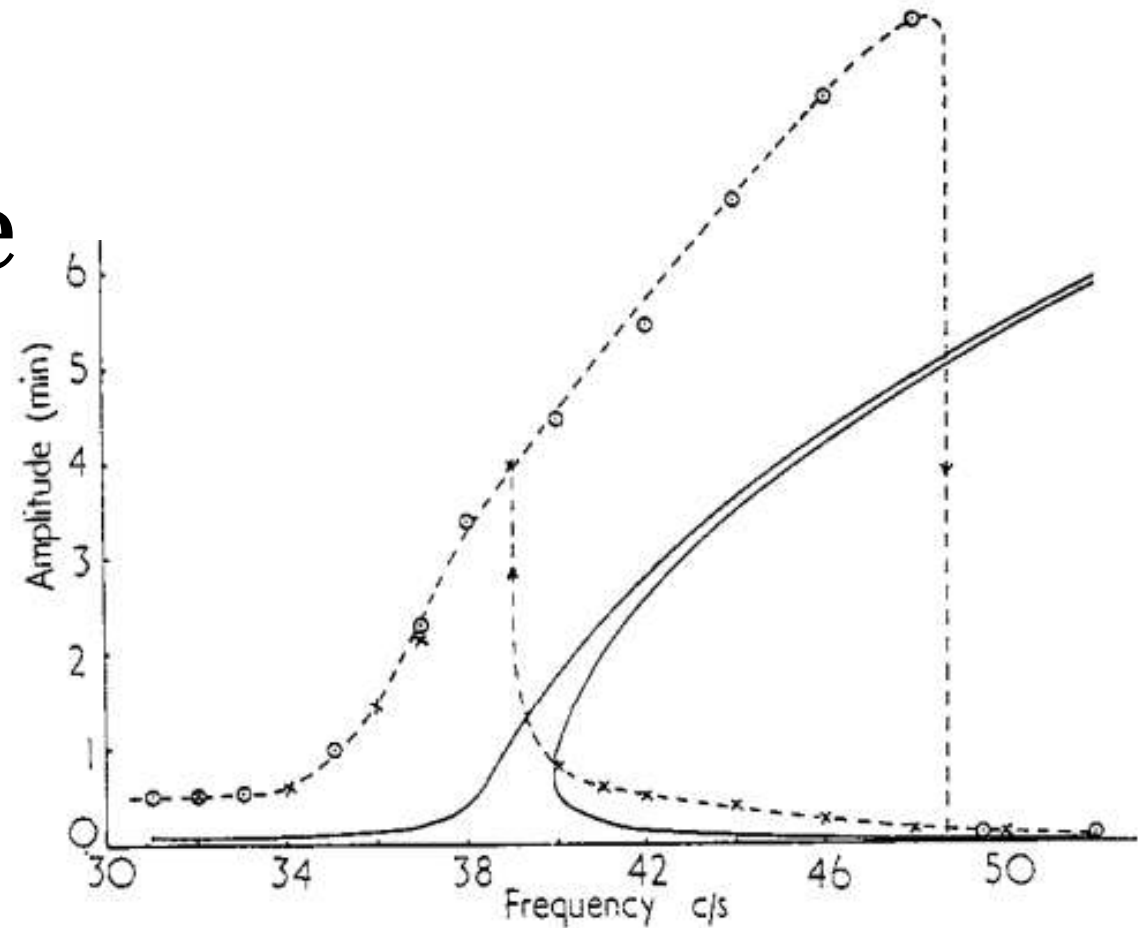
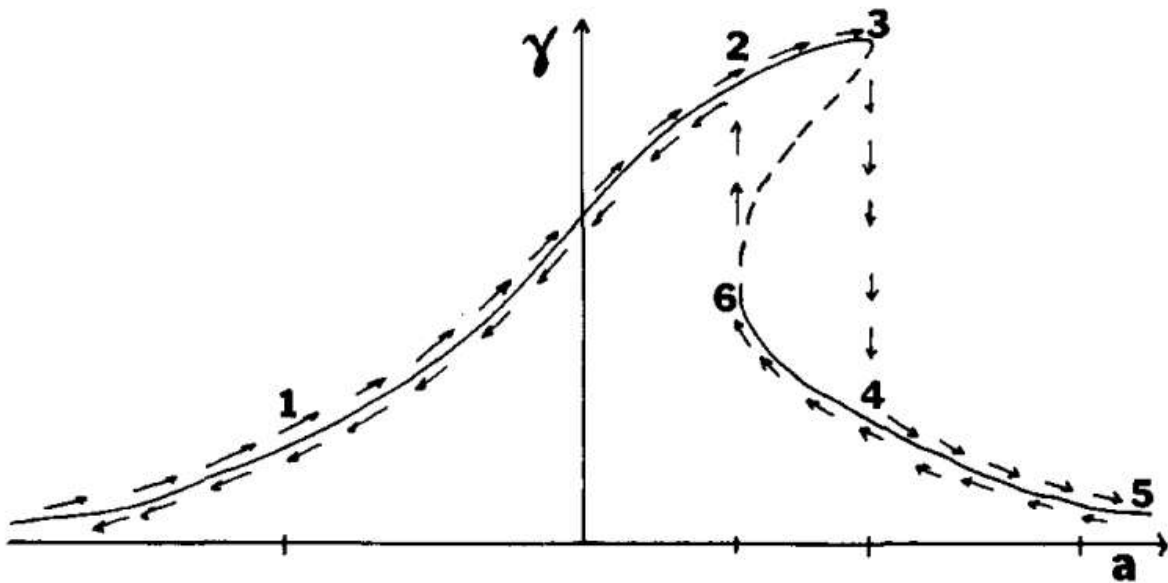
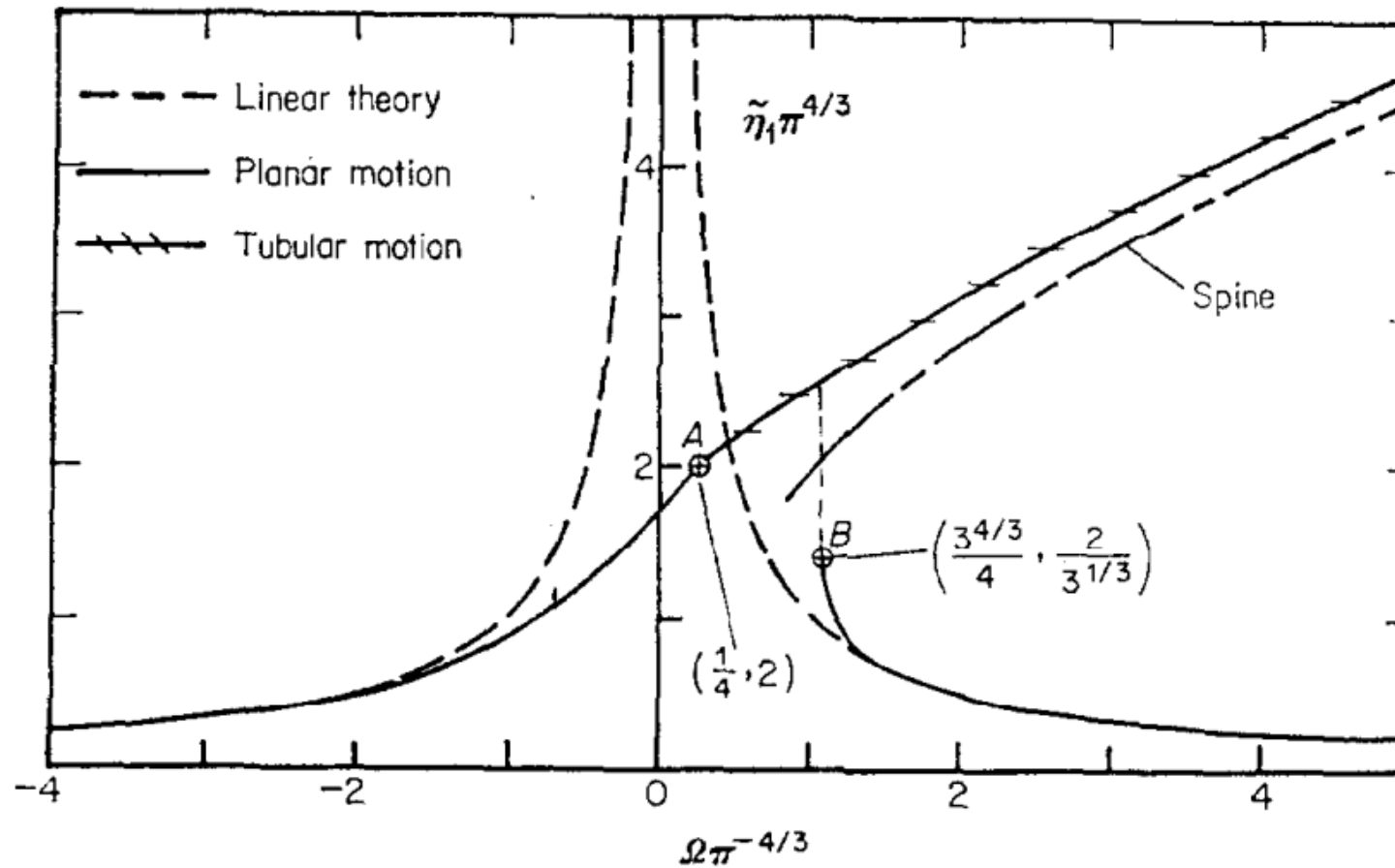


Fig. 2. Amplitude-frequency characteristics of string in its fundamental mode

- calculated from equation (5)
- - - experimental points
- ○ ○ frequency increasing
- × × × frequency decreasing

# To measure: Non-planar instability



To try:  
Non-harmonic forcing

# Chaos?

- J.M. Johnson and A.K. Bajaj, Amplitude modulated and chaotic dynamics in resonant motion of strings. Journal of Sound and Vibration (1989) 128(1), 87- 107
- N.B. Tufillaro, Nonlinear and chaotic string vibrations. Am. J. Phys. 57, 408 (1989); doi: 10.1119/1.16011
- A.K. Bajaj and J.M. Johnson, On the amplitude dynamics and crisis in resonant motion of stretched strings (1991)
- T.C. Molteno and N.B. Tufillaro, An experimental investigation into the dynamics of a string, Am. J. Phys. 72, 1157 (2004); doi: 10.1119/1.1764557
- not much done experimentally....

# Theory

- Understand some of the basics from the literature
- Compare different models to your experiments
- Find new approximations
  - preferably easier but still working

# References

- [STEM Fellowship](#)
- Sources of videos:
  - [Brotherof](#)
  - [Dan Russel](#)
  - [Prof Miller](#)
  - [SMUPhysics](#)
- Pictures:
  - [Theory](#)
- [A video series about string simulation](#)
- Scientific papers cited in respective slides

# Desmos waves superposition

<https://www.desmos.com/calculator/365anbv6mq>