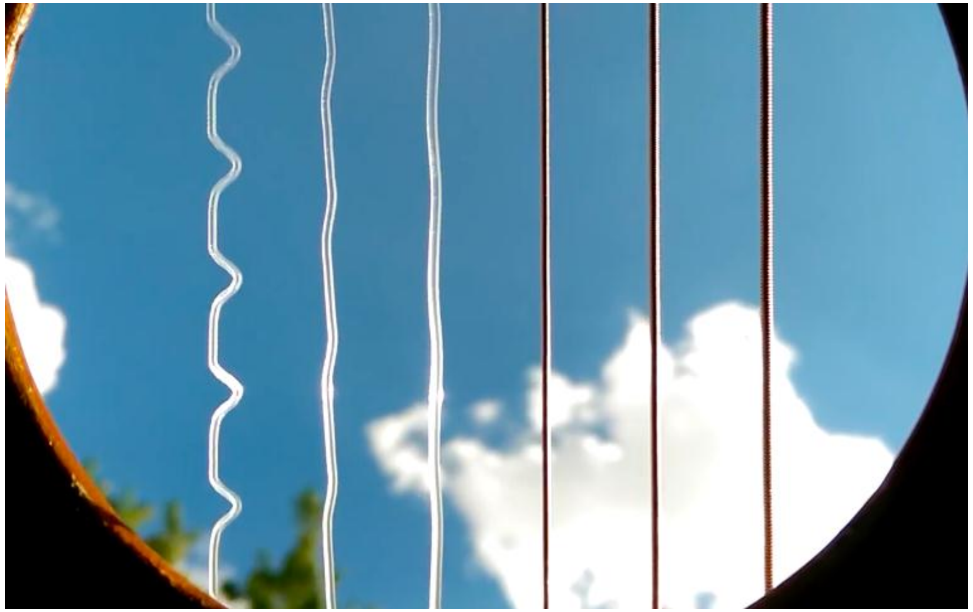
Problem No. 11
Guitar String

Ronald Dobos

12. Nov 2020



Unrelated but nice phenomenon

Links to the videos and quotes from the papers in notes ↓

Problem statement

- A **periodic** force is applied to a steel guitar string using an electromagnet. Investigate the **motion** of the guitar string **around its resonance frequency**

- Arising questions:
  - Need to be steel and guitar string? Why so?
  - How exactly forced?
  - What can be interesting about a vibrating string??

Outline

- Building intuition
  - Go through simpler cases
- Review some basic theory
- Understand its limitations
  - Finding the interesting part
- Discuss setup
- What has been done?
- What you could do

Building intuition

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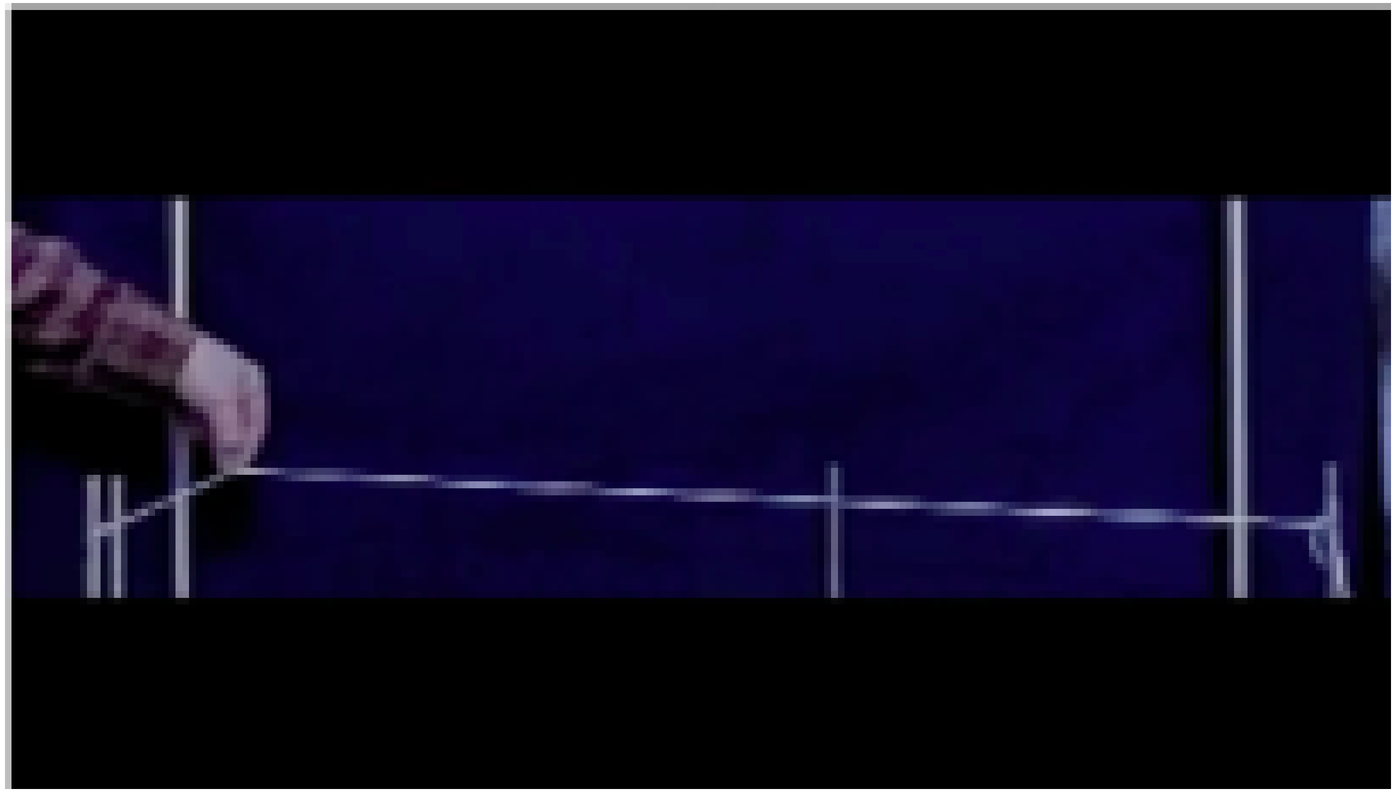
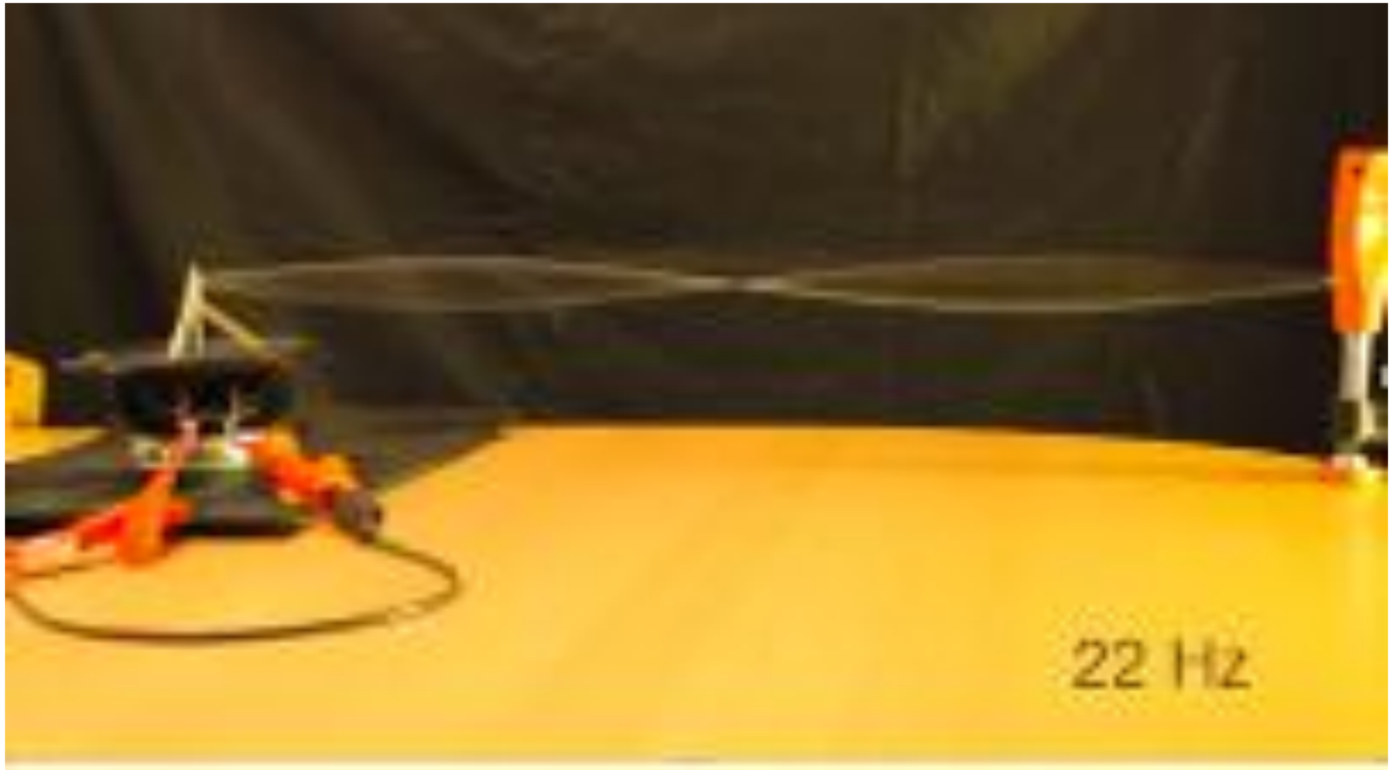
Initial disturbance

Initial disturbance

Initial disturbance



Periodic forcing



Remark: frequency has a great effect

Periodic forcing

Remark: tension has a great effect

Basic theory

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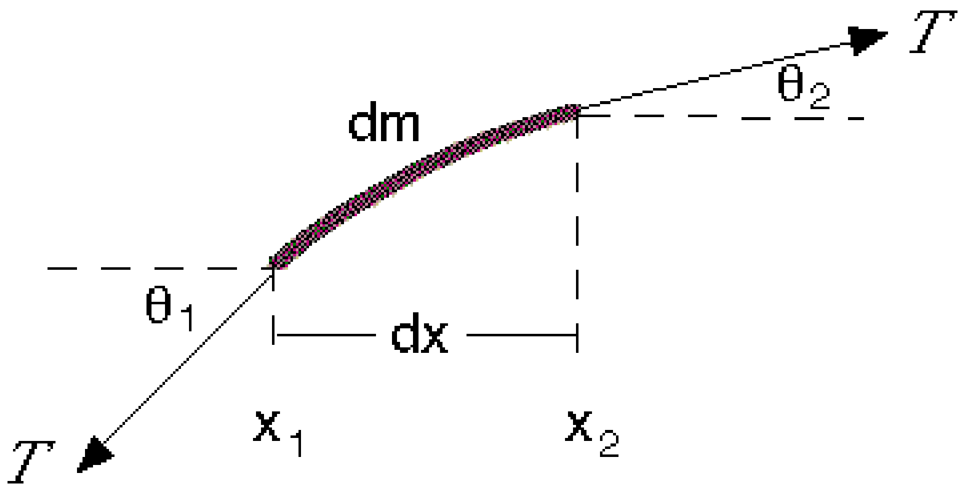
Basic theory

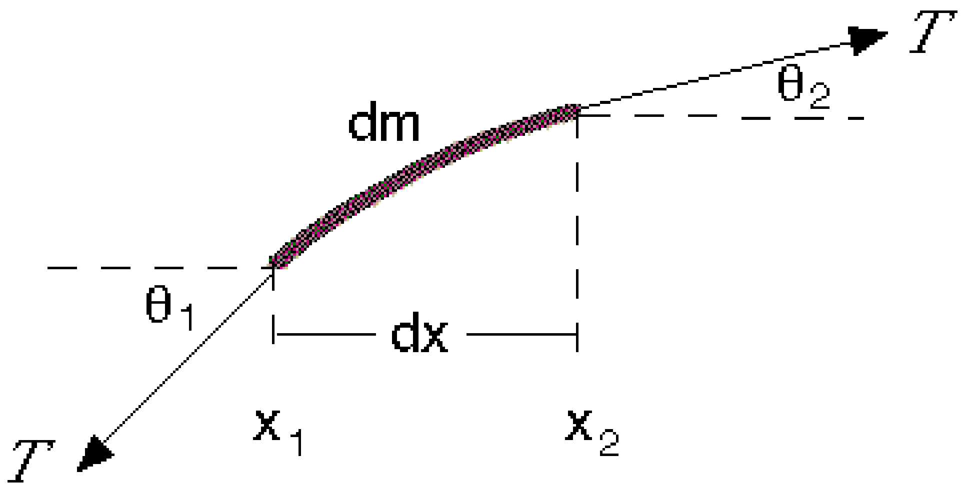
- $F_y = Tsin(\theta_2) - Tsin(\theta_1)$
- $F_y = T\left(\frac{dy(x_2)}{dx} - \frac{dy(x_1)}{dx}\right) = \mathrm{T}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}$
- $F_y = dm\frac{d^2y}{dt^2} = \mu\, dx\frac{d^2y}{dt^2}$

$$\boxed{\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2}}$$



Basic theory

$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2}$	With $\frac{T}{\mu} = c^2$



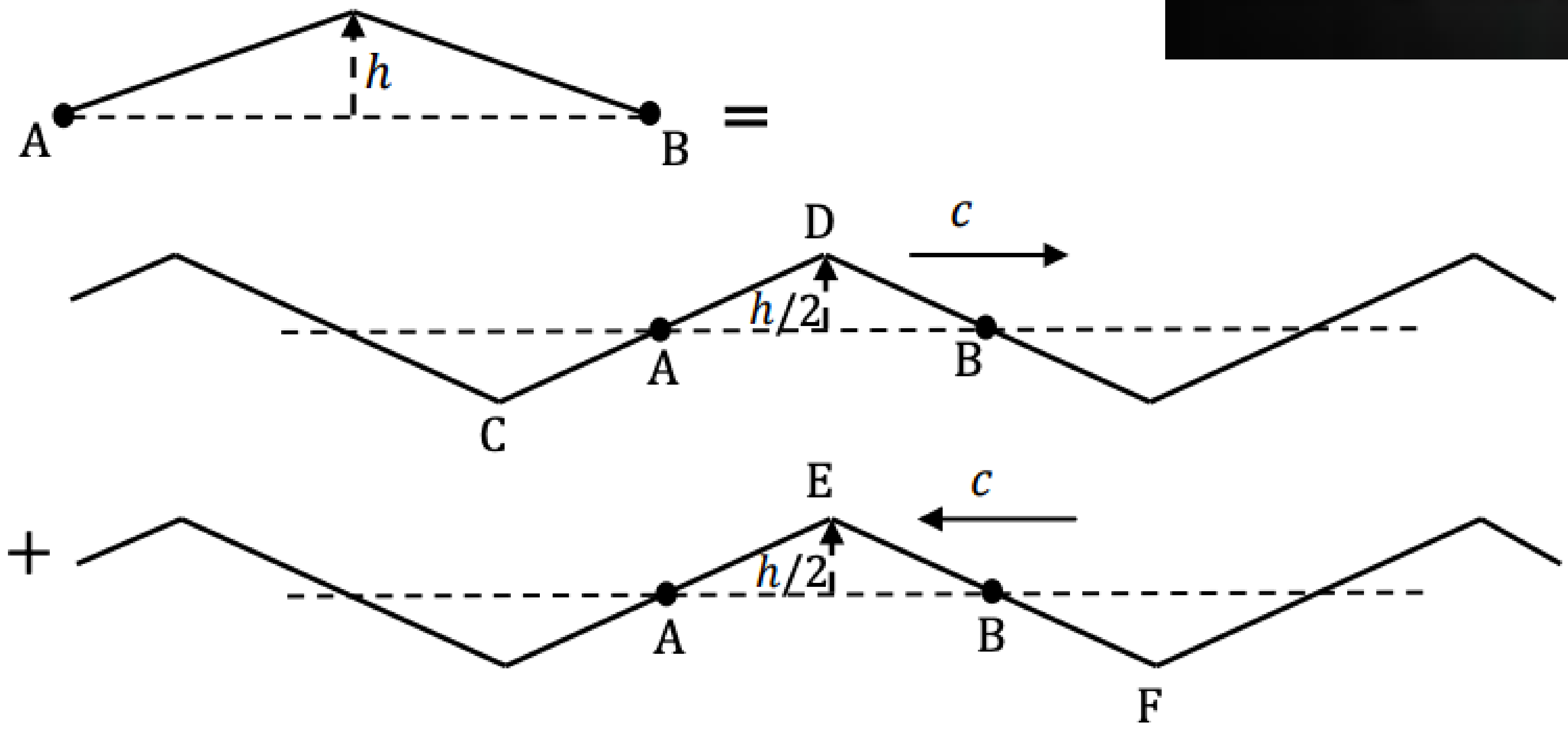
- wave equation, general solution?

$$y = A\sin(k(x - ct)) + B\cos(k(x - ct))$$?

$$y = F(x - ct) + G(x + ct)$$

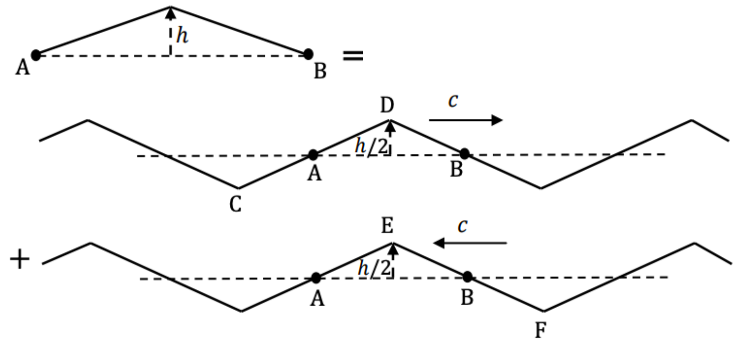
Basic theory

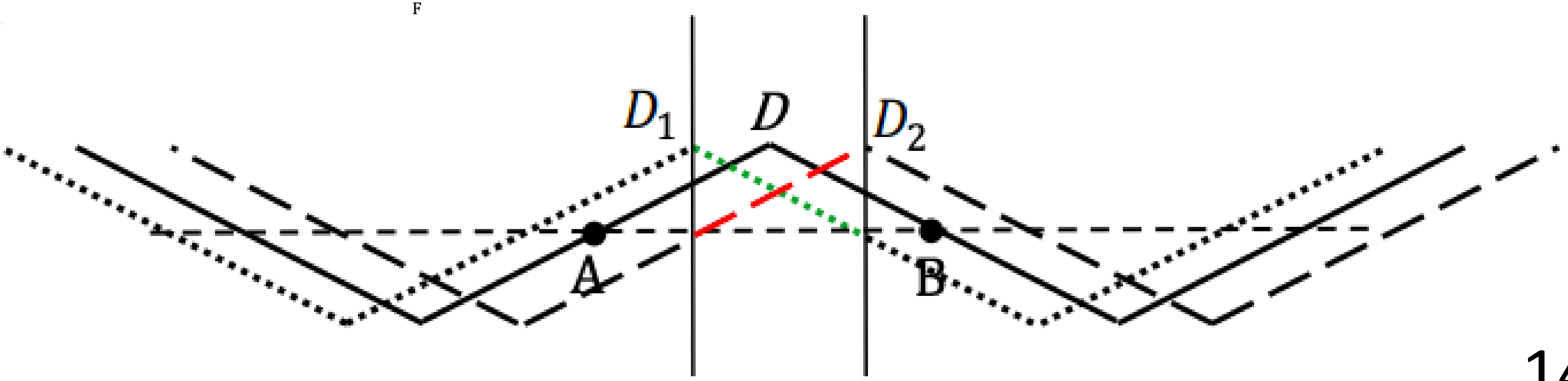
$$y = F(x - ct) + G(x + ct)$$



Basic theory

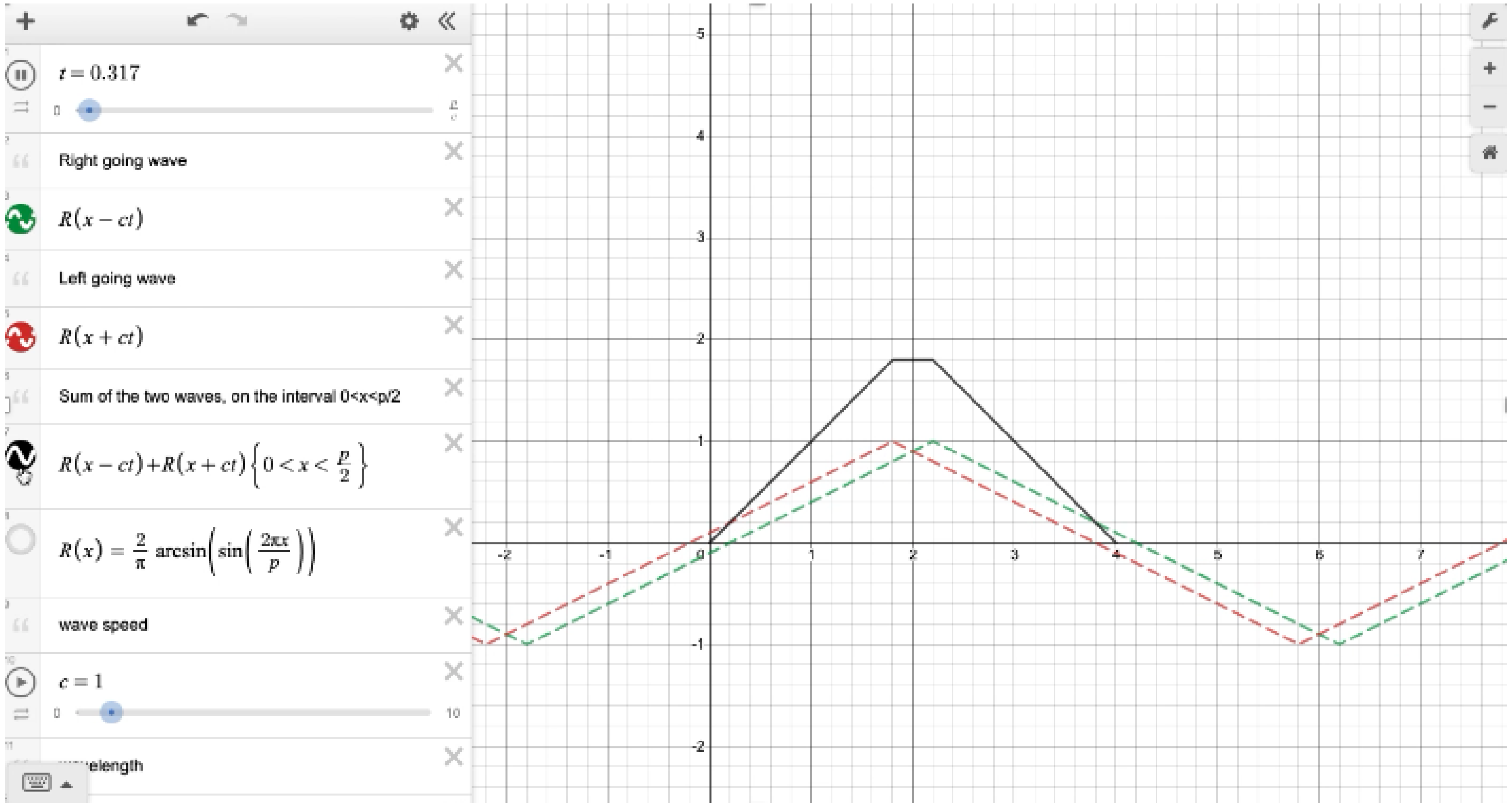
$$y = F(x - ct) + G(x + ct)$$



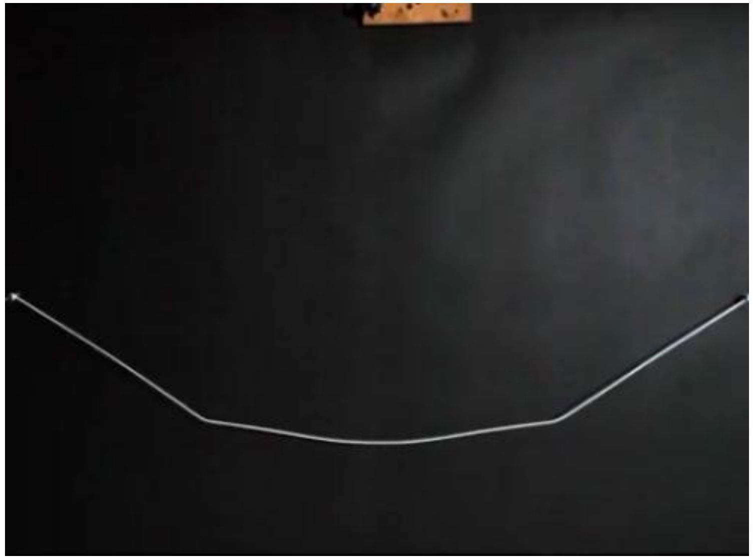


Basic theory

$$y = F(x - ct) + G(x + ct)$$



https://www.desmos.com/calculator/365anbv6mq



Basic Theory Limitations

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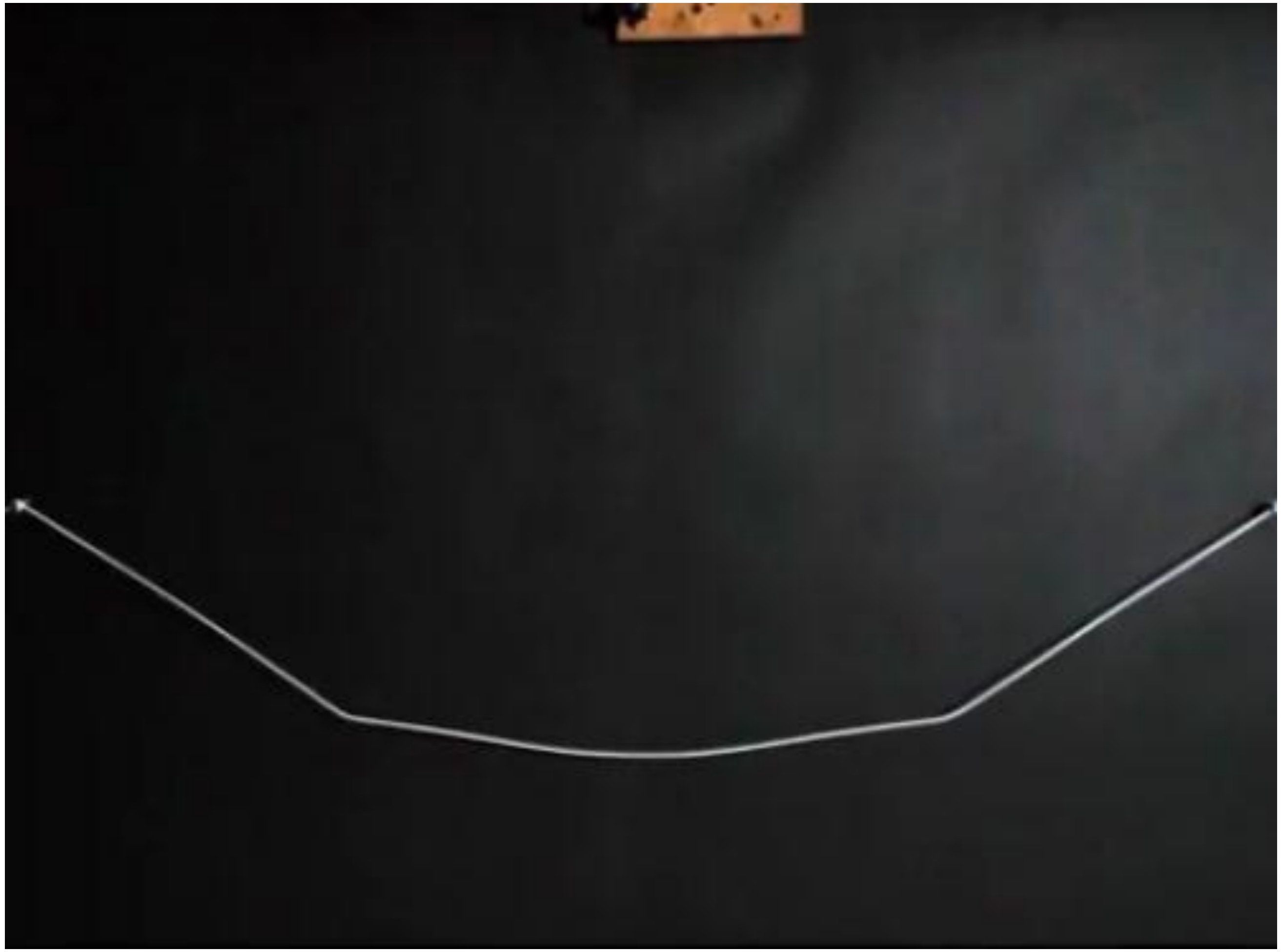
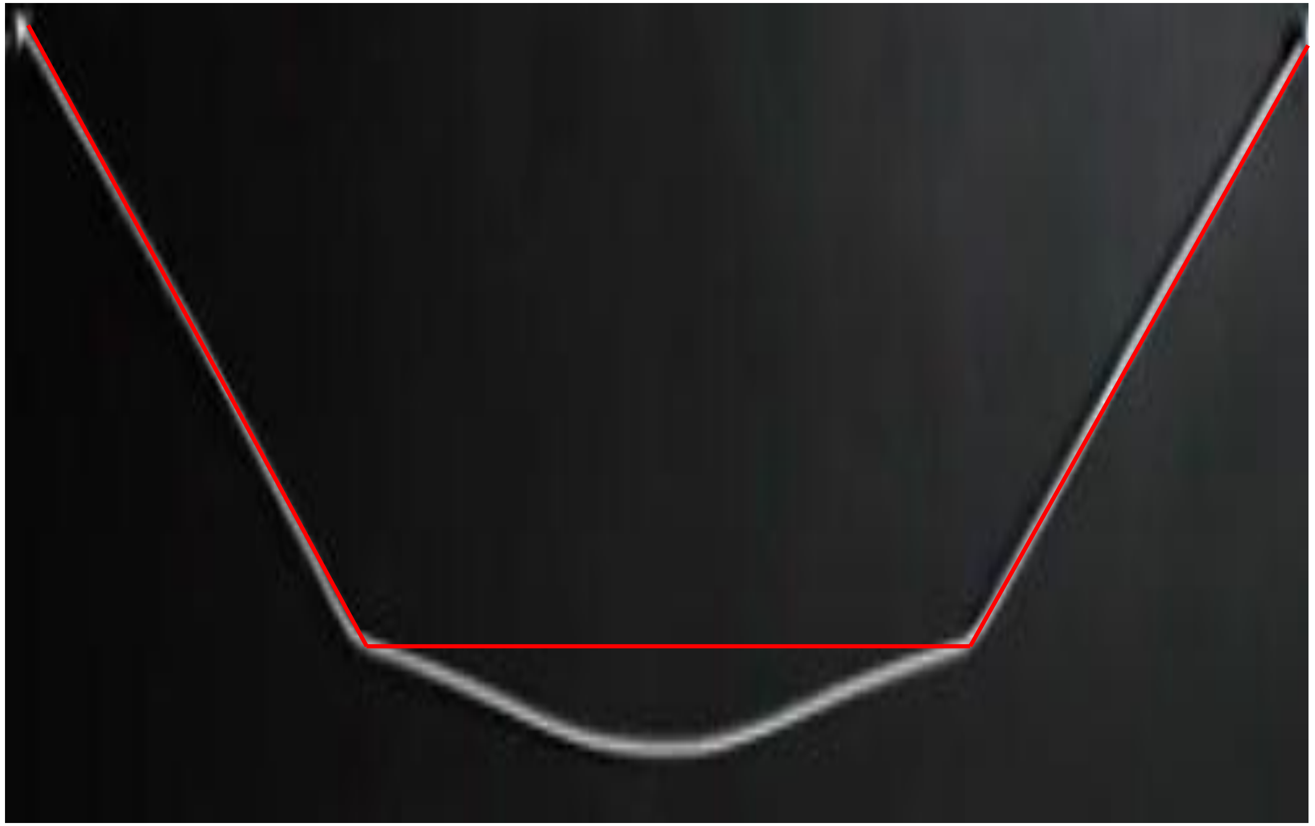
Limitation 1

$$y = F(x - ct) + G(x + ct)$$



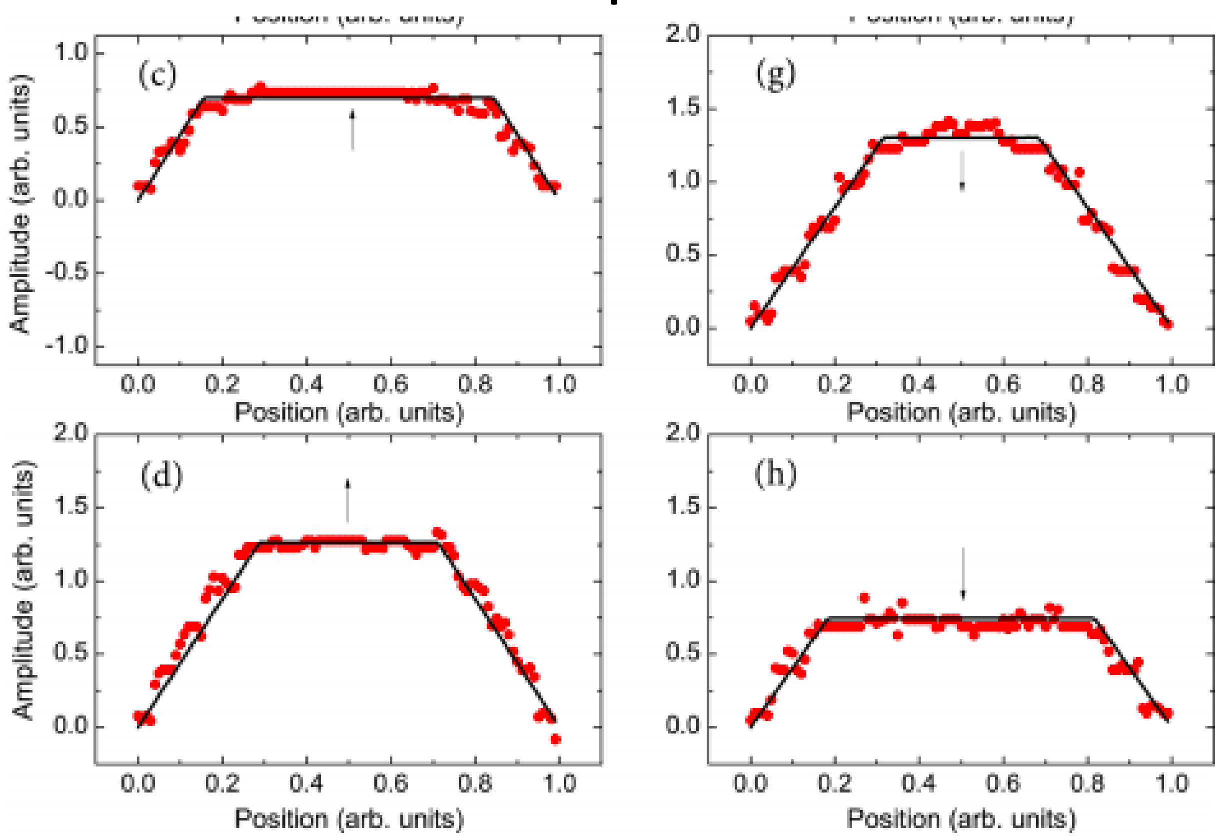
Limitation 1

$$y = F(x - ct) + G(x + ct)$$

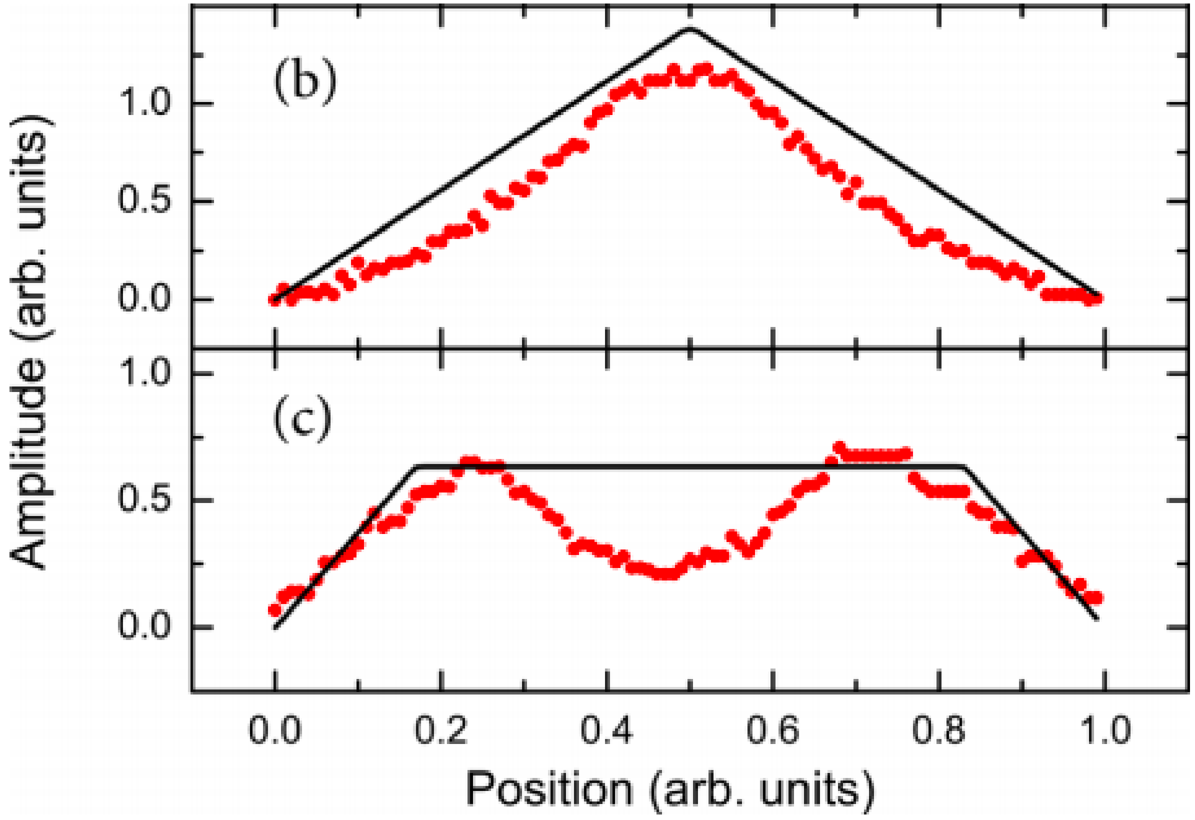


Limitation 1

Small amplitudes



Big amplitudes



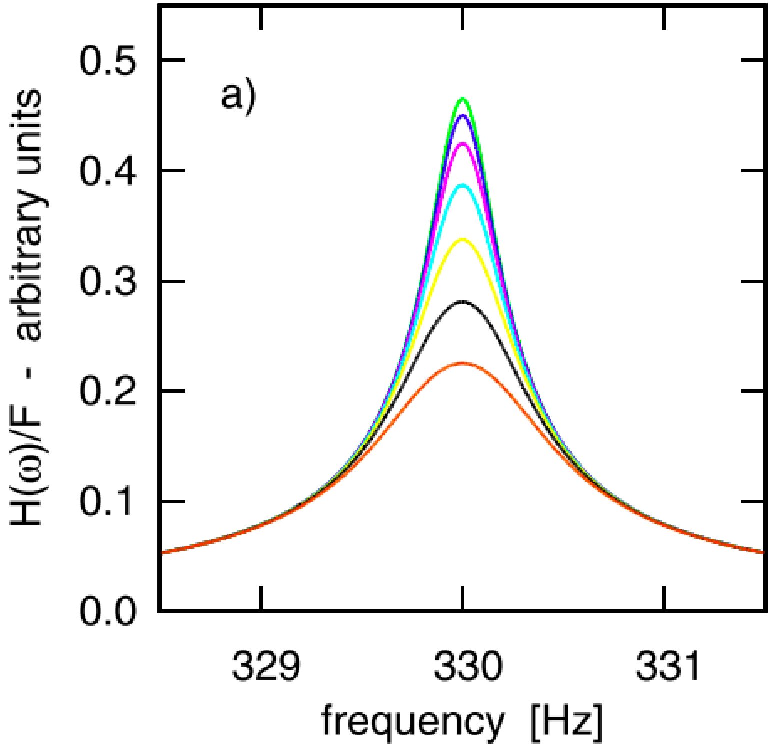
S. B. Whitfield and K. B. Flesch, An experimental analysis of a vibrating guitar string using high-speed photography
doi: 10.1119/1.4832195

Limitation 2

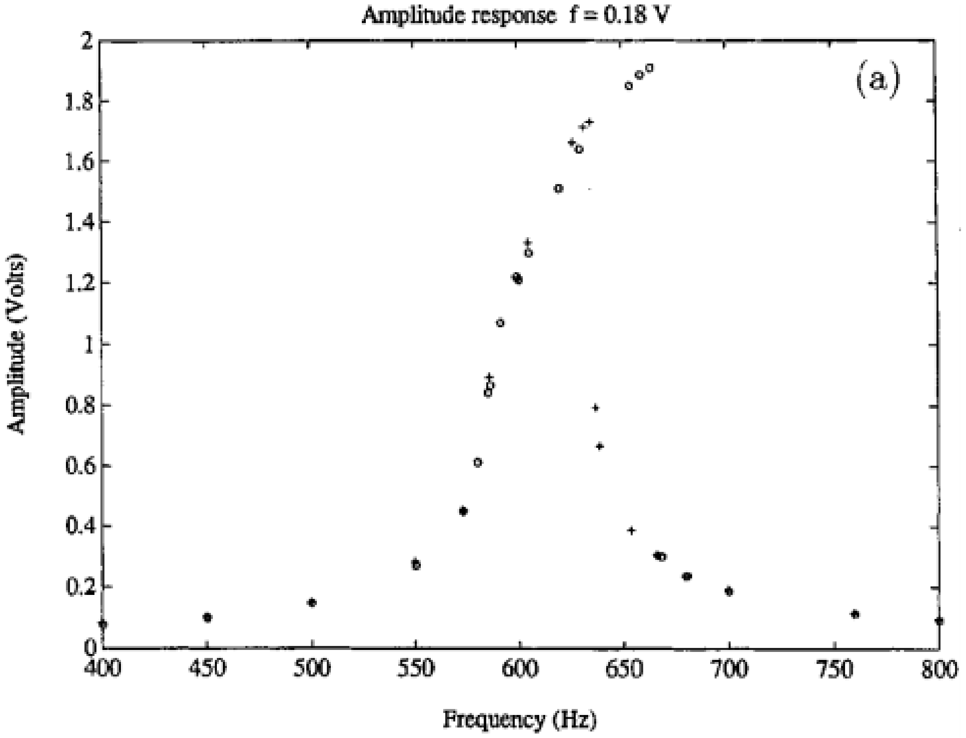
$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} + \mathrm{F(x,t)}$$ With $\frac{T}{\mu} = c^2$

- Considering an external periodic forcing $F(x,t)$

Expectation



Reality

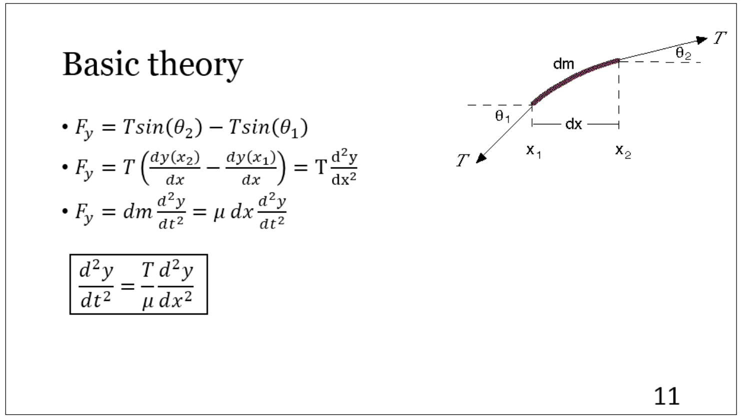


M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)
T. C. Molteno, An experimental investigation into the dynamics of a string. AJP (2004)

Basic theory

What went wrong?

…



…

We assumed some things

- Constant tension
- Small amplitudes
- Planar motion

Past literature review

Problem No. 11 Guitar string

Oplinger 1960

“Modified wave equation for transverse motion”:

$$\frac{d^2y}{dt^2} = \frac{T_0}{\mu}\frac{d^2y}{dx^2} + \frac{1}{2}\frac{\mathrm{EA}}{\mu}\left[\frac{1}{\mathrm{L}}\int_0^L \left(\frac{dy}{dx}\right)^2 dx\right]\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2}$$

Accounts for tension changes

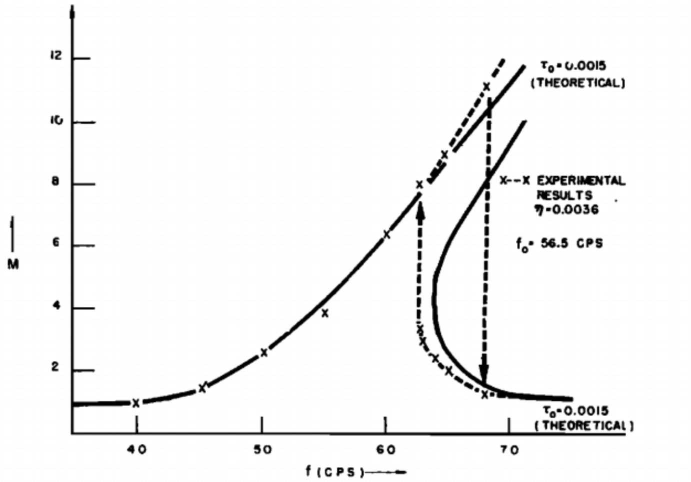


FIG. 10. Comparison of theoretical and experimental results for a typical case. Note the presence of observed jump frequencies.

Oplinger, D. W. (1960). *Frequency Response of a Nonlinear Stretched String. (1960)* doi:10.1121/1.1907948

Oplinger 1960

“Modified wave equation for transverse motion”:

$$\frac{d^2y}{dt^2} = \frac{T_0}{\mu}\frac{d^2y}{dx^2} + \frac{1}{2}\frac{\mathrm{EA}}{\mu}\left[\frac{1}{\mathrm{L}}\int_0^L \left(\frac{dy}{dx}\right)^2 dx\right]\frac{d^2y}{dx^2}$$

Accounts for tension changes

Only one transverse direction

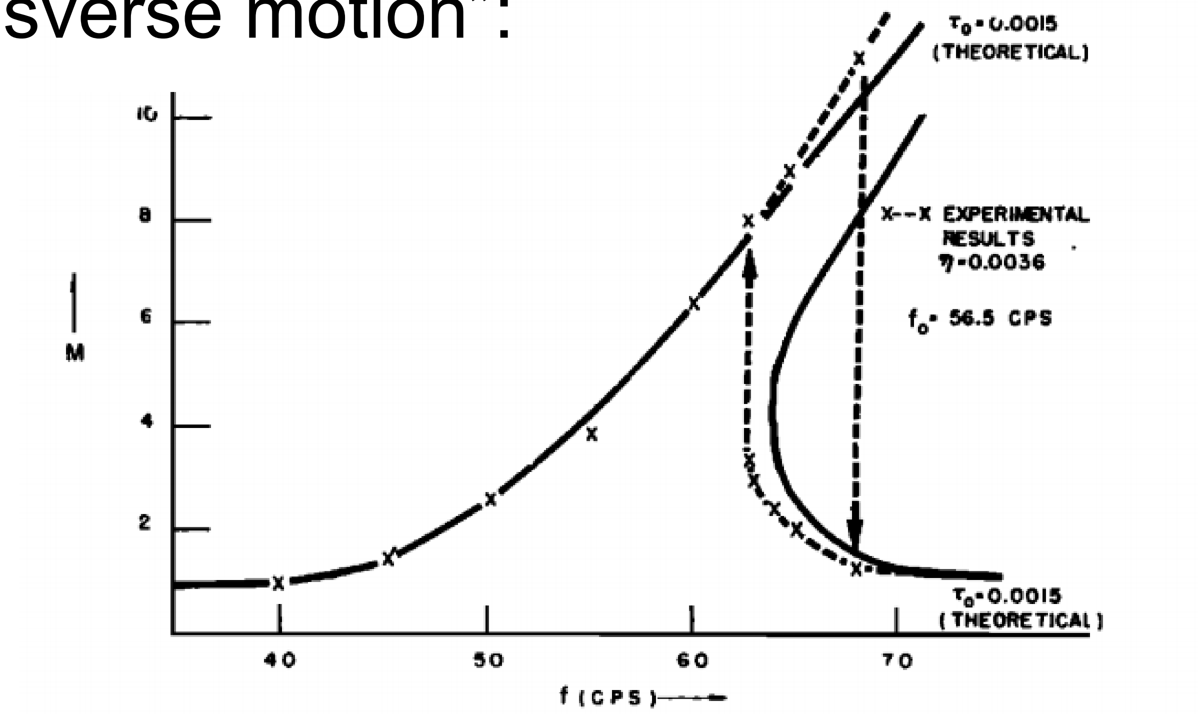


FIG. 10. Comparison of theoretical and experimental results for a typical case. Note the presence of observed jump frequencies.

Oplinger 1960

- ”For these and other curves it was found convenient to write a program for a Bendix G-15 computer”

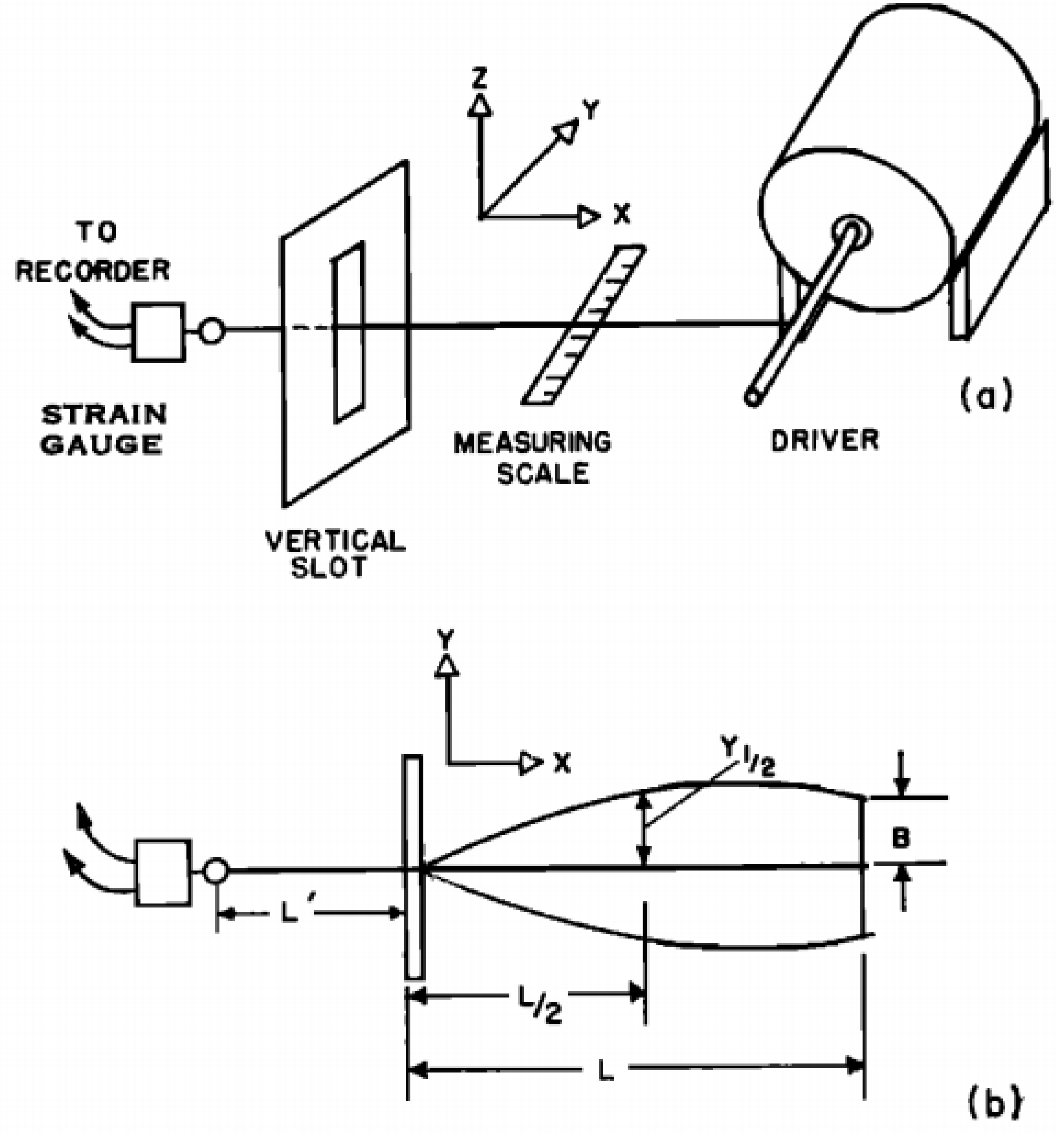


FIG. 9. (a) Experimental apparatus; (b) basic parameters.

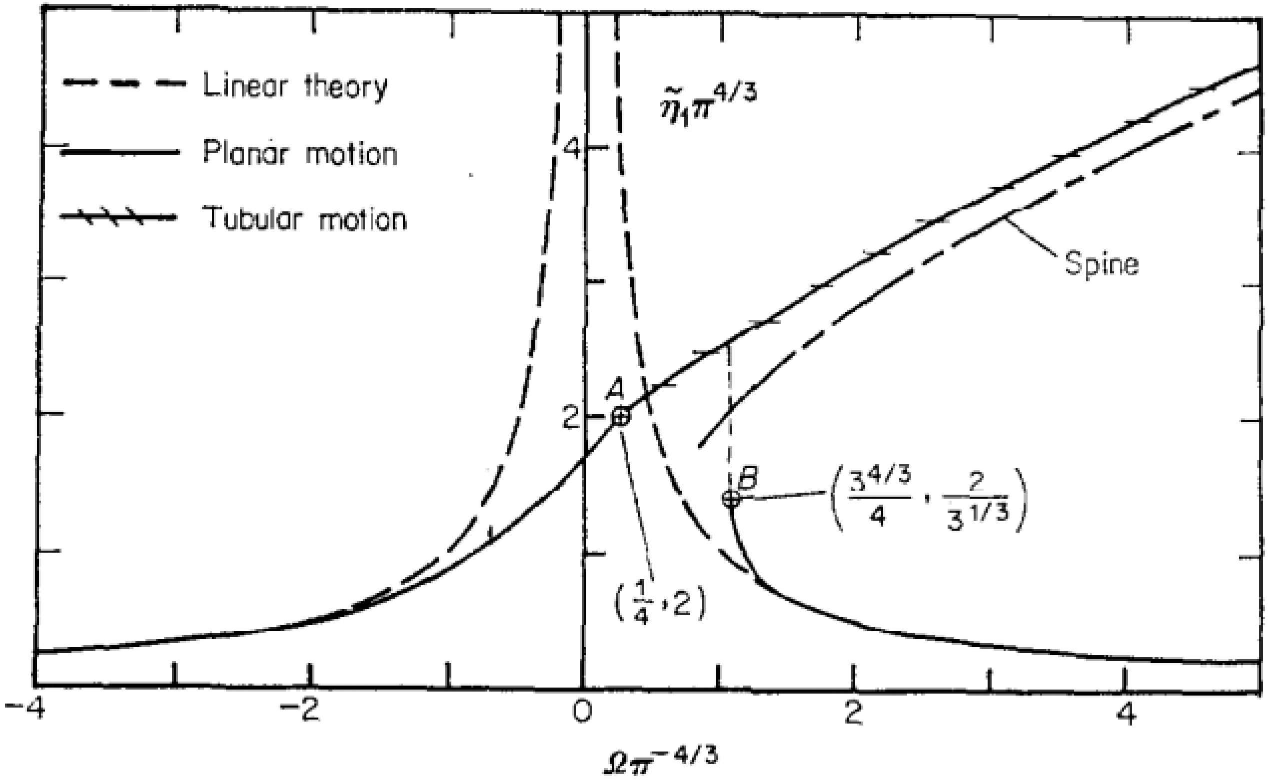
Narasimha 1968

$$(1 - u_x)\,\ddot{u} = c_1^2 \frac{\partial \Lambda}{\partial x} + h,$$

$$(1 - u_x)\,\ddot{\mathbf{v}} = c_1^2 \frac{\partial}{\partial x}(\mathbf{v}_x \Lambda) + \mathbf{f} + \mathbf{g},$$

$$\Lambda \equiv \frac{1 + c_1^2\lambda - c_2^2\lambda^2 + c_3^2\lambda^3}{c_1^2(1+\lambda)(1-u_x)},$$

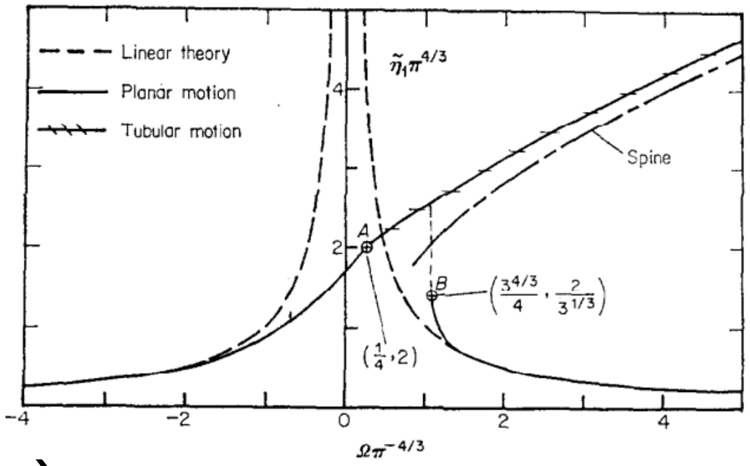
- Found the exact equations for 3D motion.



R. Narasimha, Non-linear vibration of an elastic string. J. Sound Vib (1968) 8 (1), 134-146

Narasimha 1968

- Non-planar oscillations are parametrically excited by planar oscillations (found Mathieu eq.)



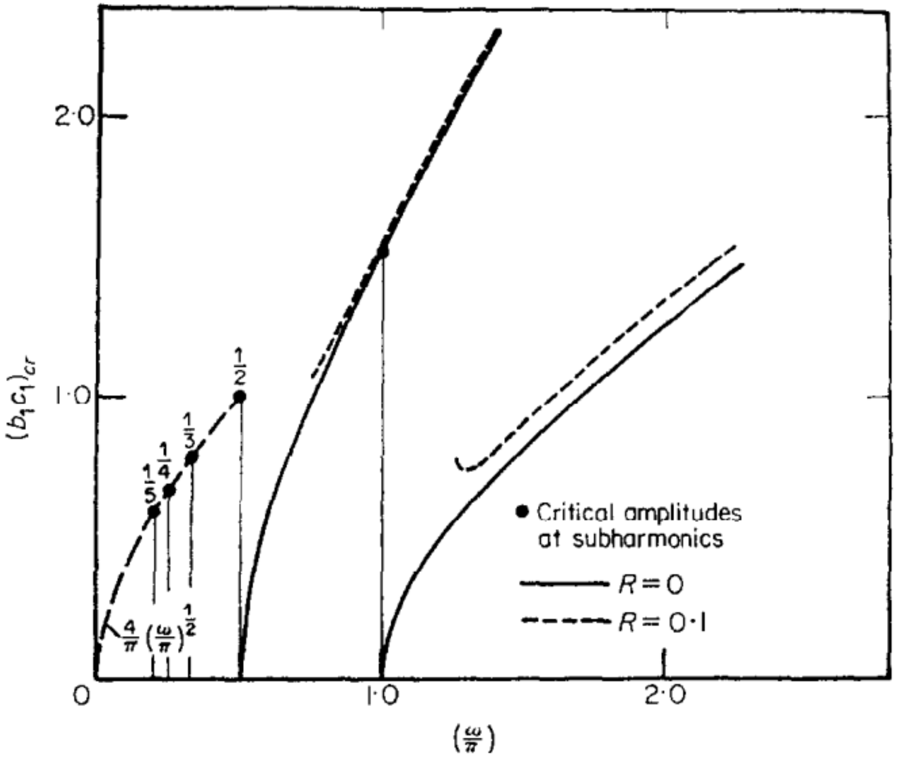


Figure 4. Critical amplitudes for onset of a second mode of transverse motion. With no damping ($R = 0$), the critical amplitude generally increases with frequency, but drops suddenly to zero at the natural frequency and all subharmonics. The largest critical amplitudes necessary for instability occur at frequencies just below these, and are shown by full circles, with numbers to indicate the subharmonic. The dashed line through the origin is the locus of these peak amplitudes and has the asymptotic form $(b_1 c_1)_{cr} \approx 4\omega^{1/2}\pi^{-3/2}$ as $\omega \to 0$.

R. Narasimha, Non-linear vibration of an elastic string. J. Sound Vib (1968) 8 (1), 134-146

Elliot 1980

Go general, then approximate down.

Find energy in terms of the string shape:

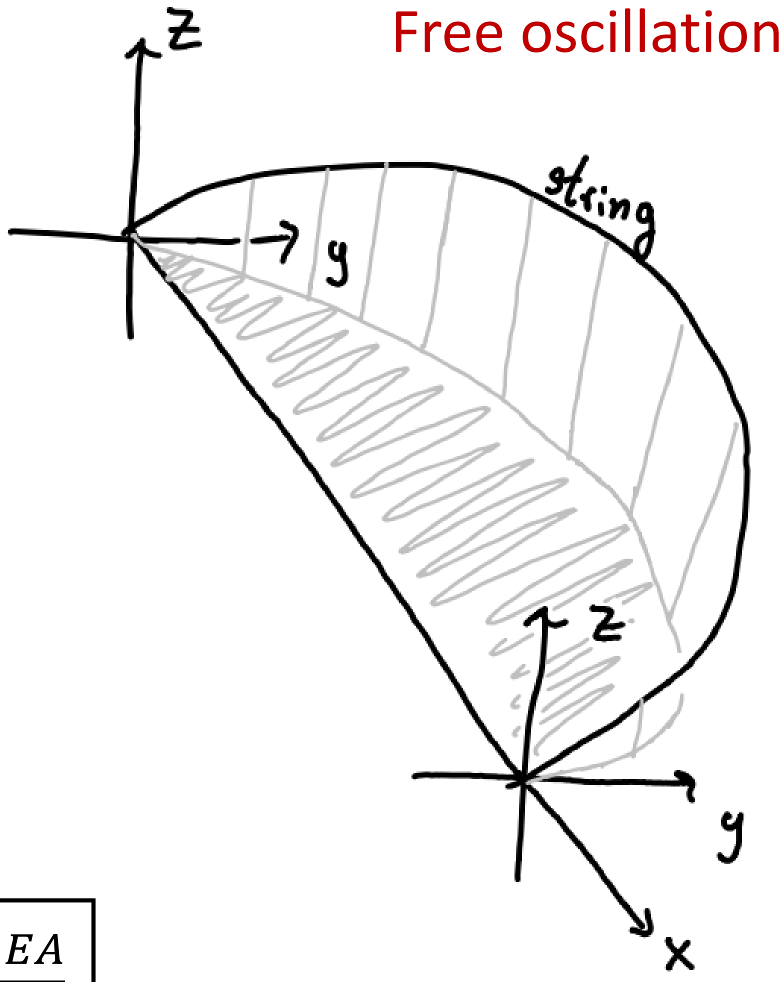
$$\begin{pmatrix} y(x,t) \\ z(x,t) \end{pmatrix} = \sum_j \begin{pmatrix} V_{xj} \\ V_{yj} \end{pmatrix} \sin\left(\frac{j\pi x}{L}\right)$$

now plug into energy equation and obtain:

$$\ddot{Y} + \omega^2 Y[1+\sigma(Y^2+Z^2)] = 0$$
$$\ddot{Z} + \omega^2 Z[1+\sigma(Y^2+Z^2)] = 0$$

$$\omega = \frac{T_0}{2\mu L}, \quad \sigma = \frac{3EA}{8T_0}$$

F represents the first Fourier component of force



Free oscillations!

Elliot 1980

Taking limit of $Z \ll Y$

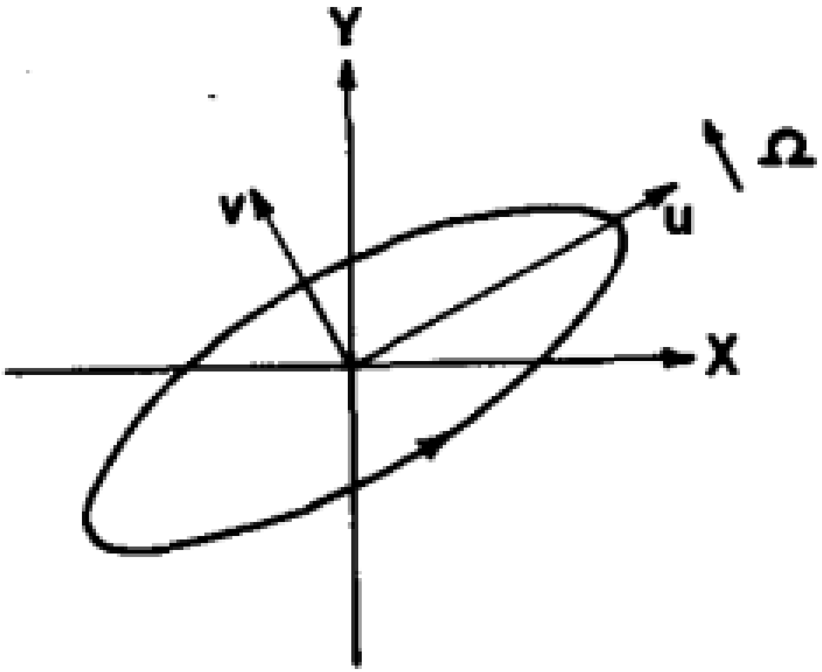
$$\ddot{Y} + \omega^2 Y[1 + \sigma Y^2] = 0$$
$$\ddot{Z} + \omega^2 Z[1 + \sigma Y^2] = 0$$

Got Duffing eq. in Y solved (approx.) by:

$$y = a\cos(pt) \qquad p = \omega(1 + \sigma a^2/2)$$

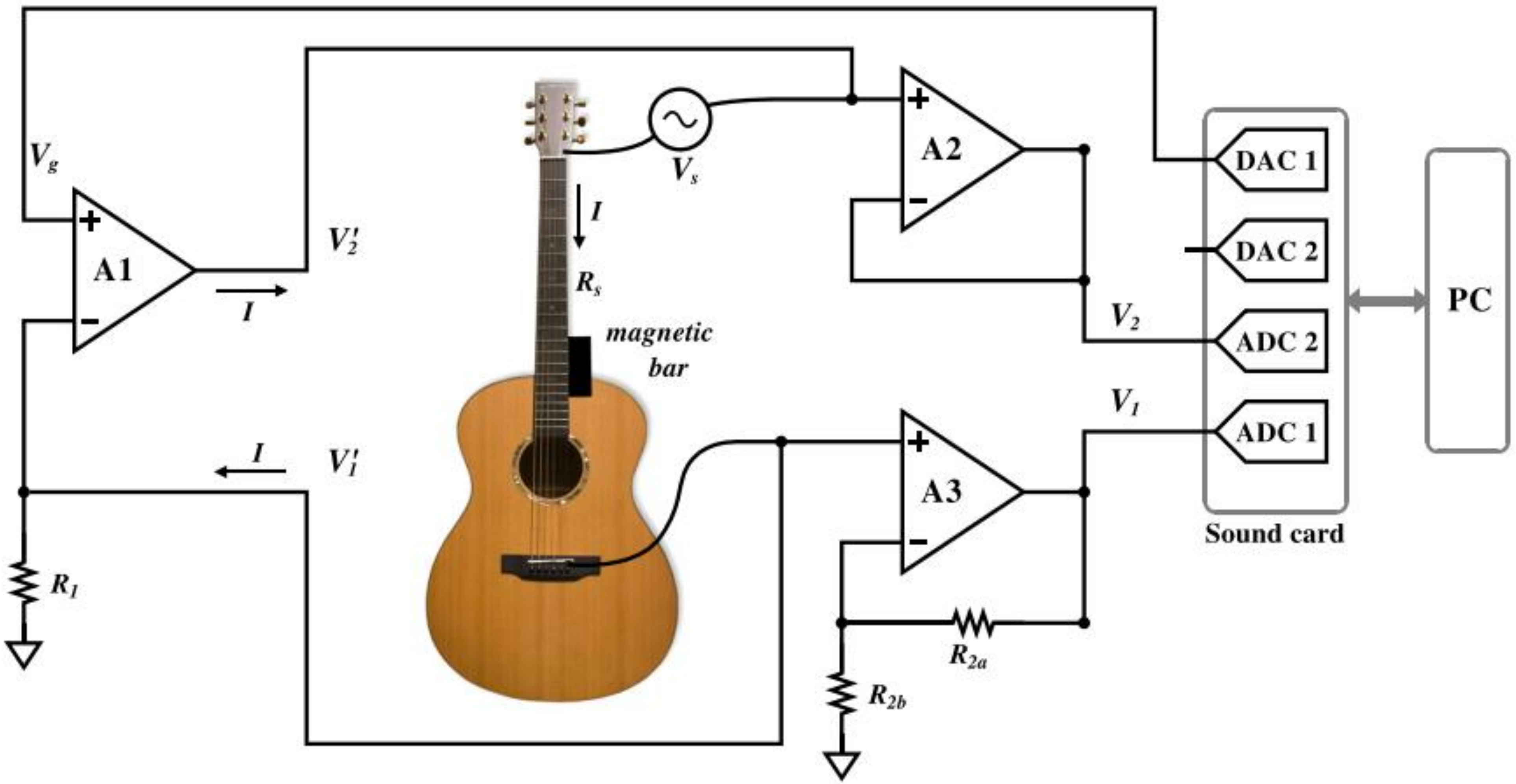
Then in the other direction have:

$$\ddot{Z} + \omega^2 Z\left[1 + \frac{\sigma a^2}{2}(1 + \cos(2pt))\right] = 0$$



J.A. Elliot, Intrinsic nonlinear effects in vibrating strings. Am. J. Phys. 48, 478 (1980)

Carla 2017

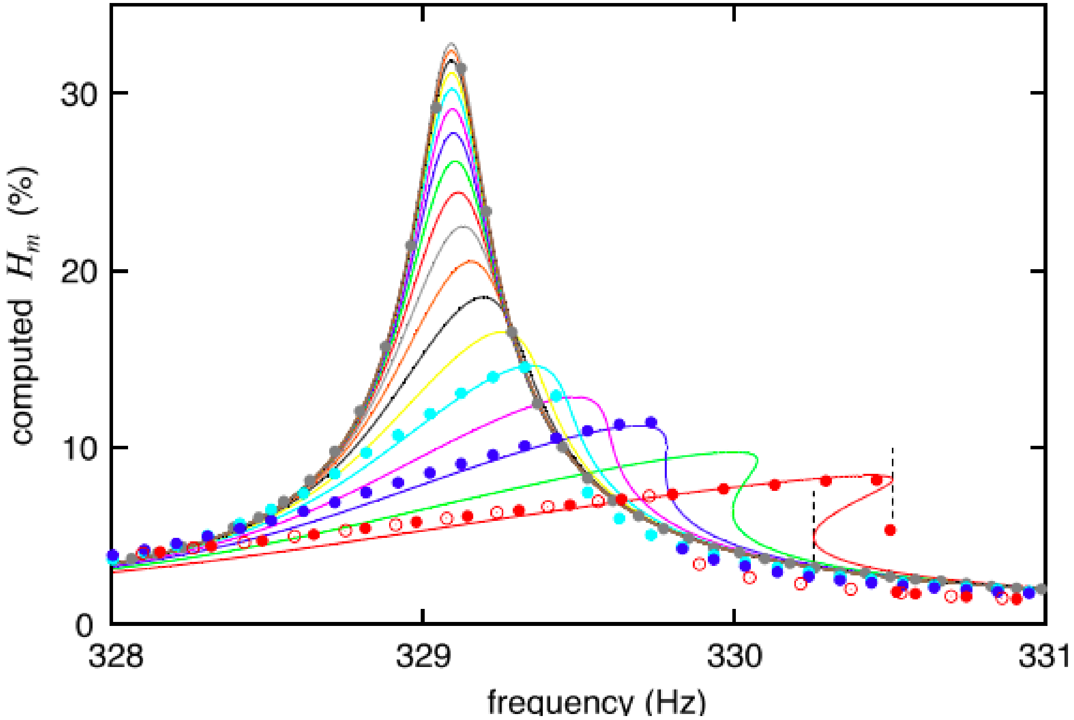


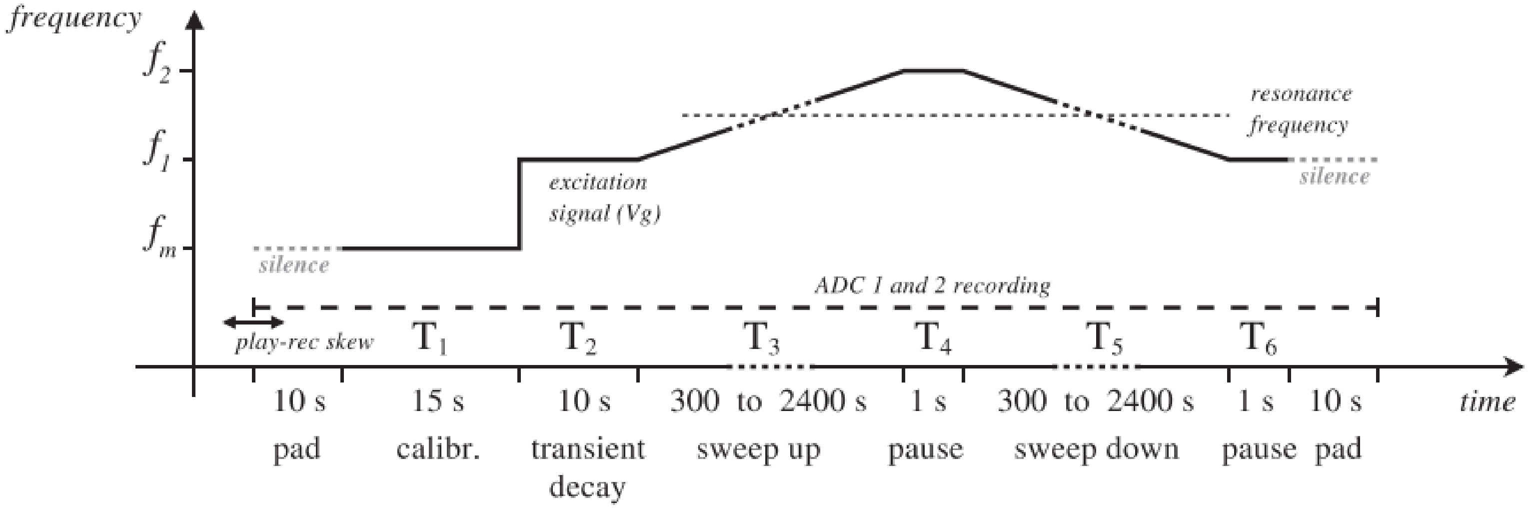
M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

Carla 2017

$$\ddot{Y} + \omega\beta\dot{Y} + \omega^2 Y[1 + \sigma Y^2] = 0$$

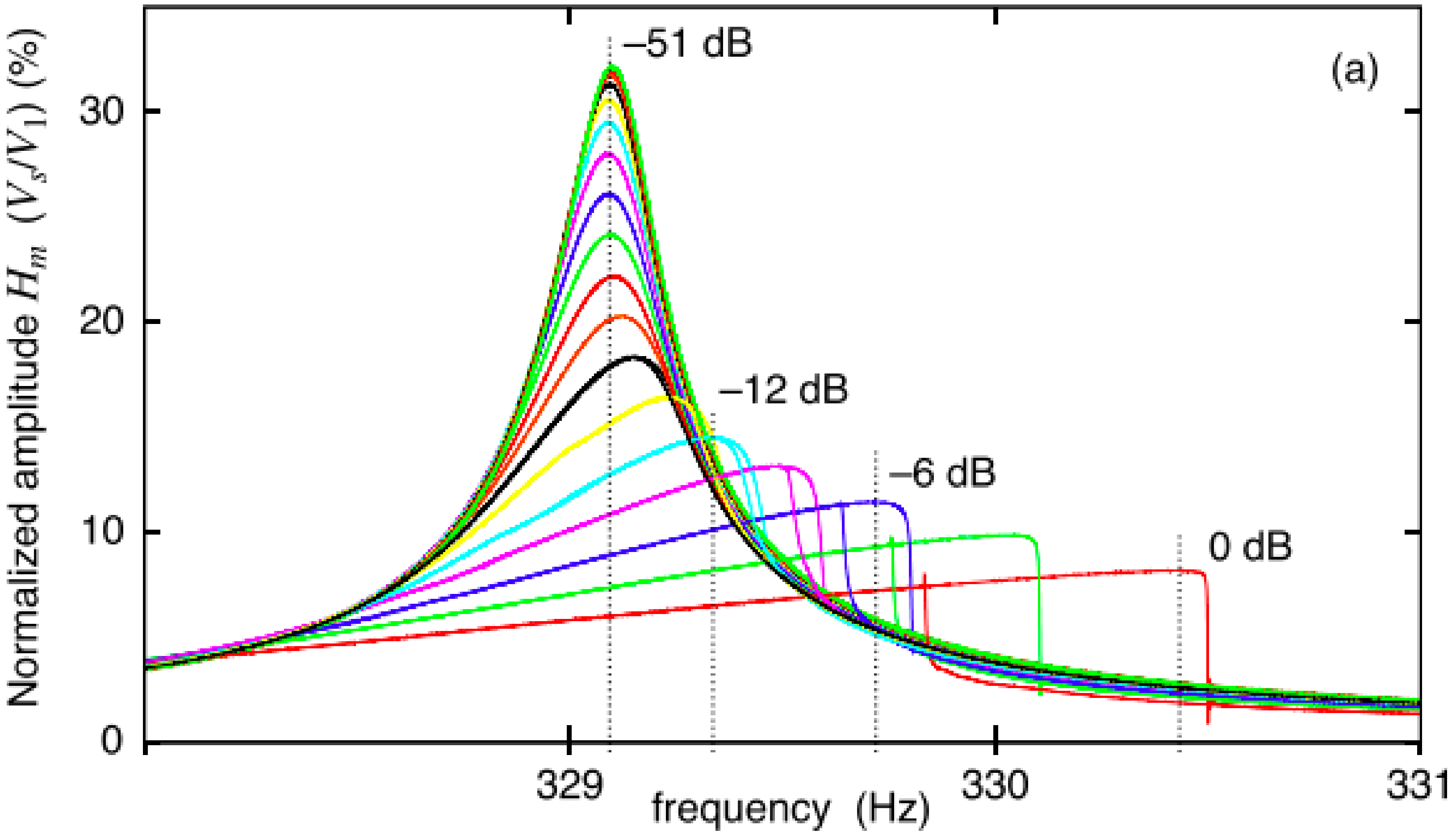
Comparing experiments with the damped Duffing equation

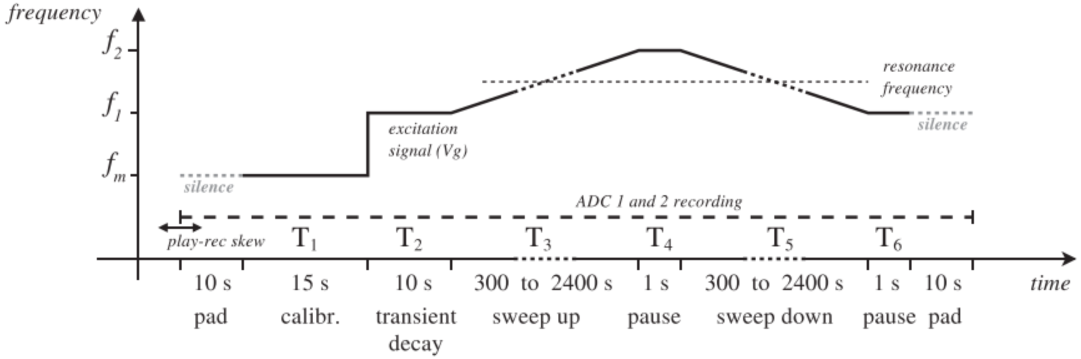




M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

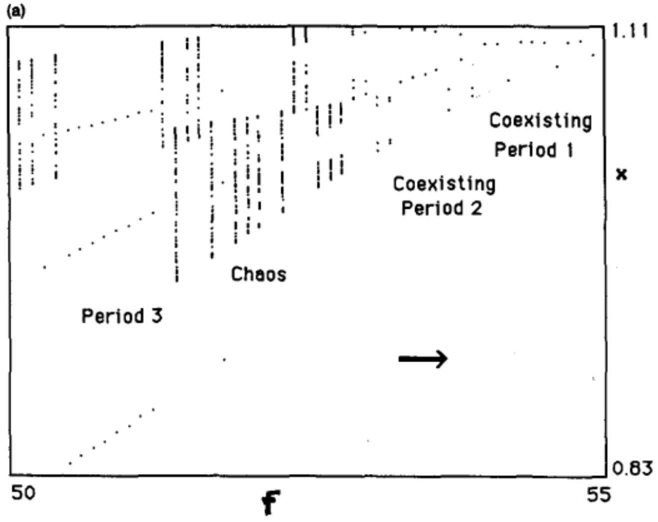
Carla 2017

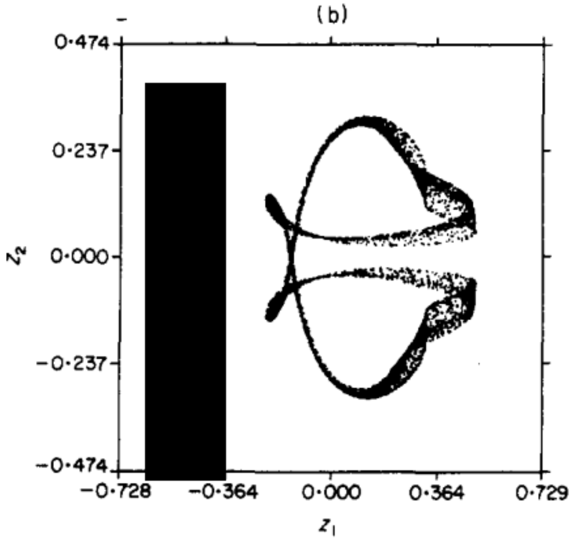


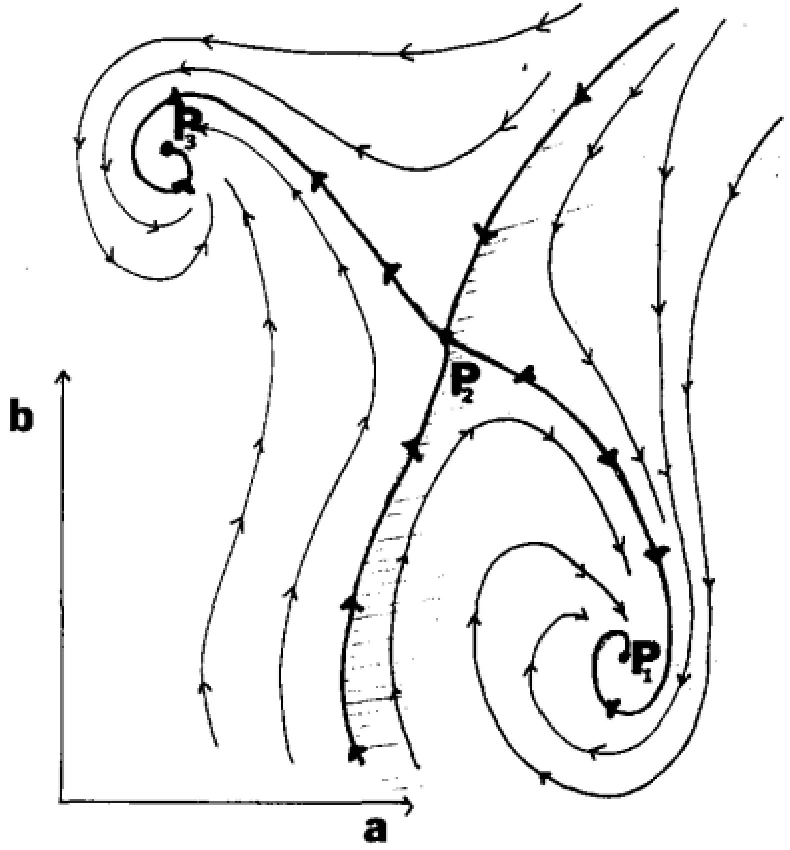


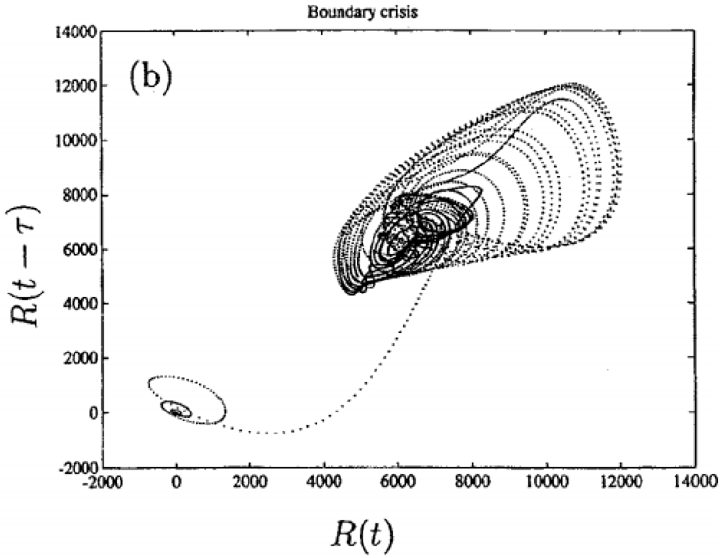
M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

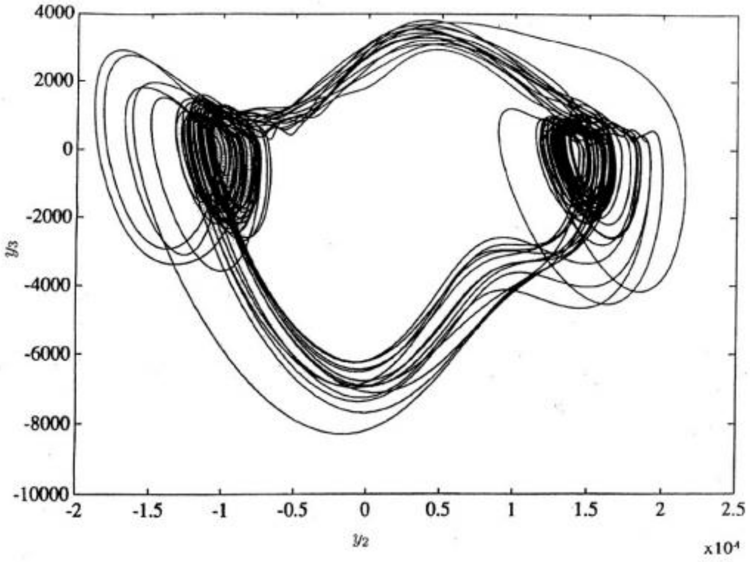
And a lot of chaos theory…











Setup

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Setup - forcing

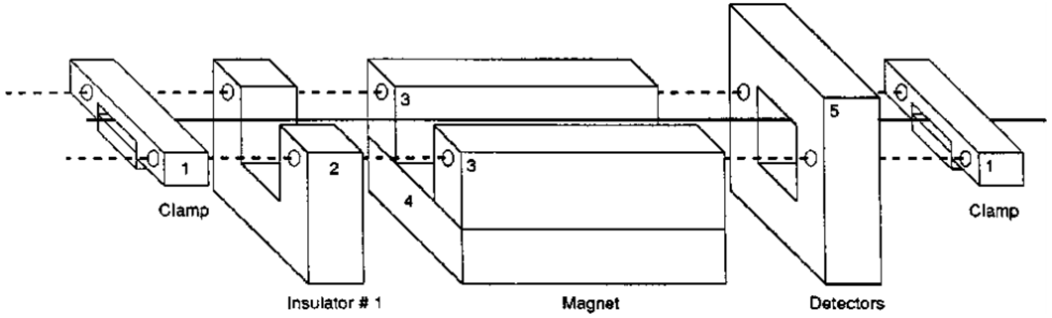


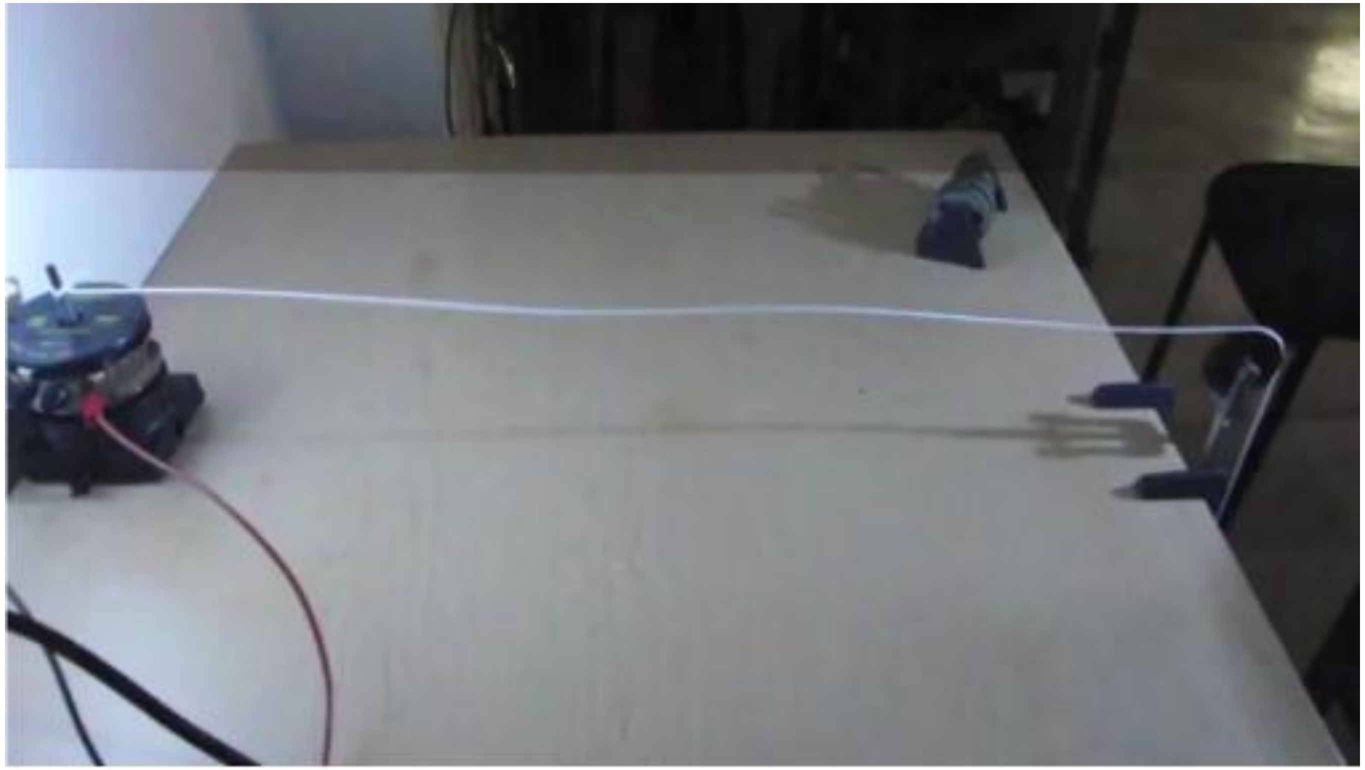
Fig. 1. Schematic of the string mounting system.

- Most of modern literature uses alternating current in string + a magnet to force the string
  - How does this fit the pr. statement?
  - Use non-ferromagnetic wire for non-planar motions
- Could also use ferromagnetic string and oscillating magnetic field from an electromagnet
- A mechanical forcing should reproduce the effects (see ref.), but violates pr. statement (?)

T. C. Molteno N.B. Tufillaro, An experimental investigation into the dynamics of a string. doi: 10.1119/1.1764557

Setup – detection 1

- Use stroboscope to “freeze vibrations”
  - Then capture with a camera



M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

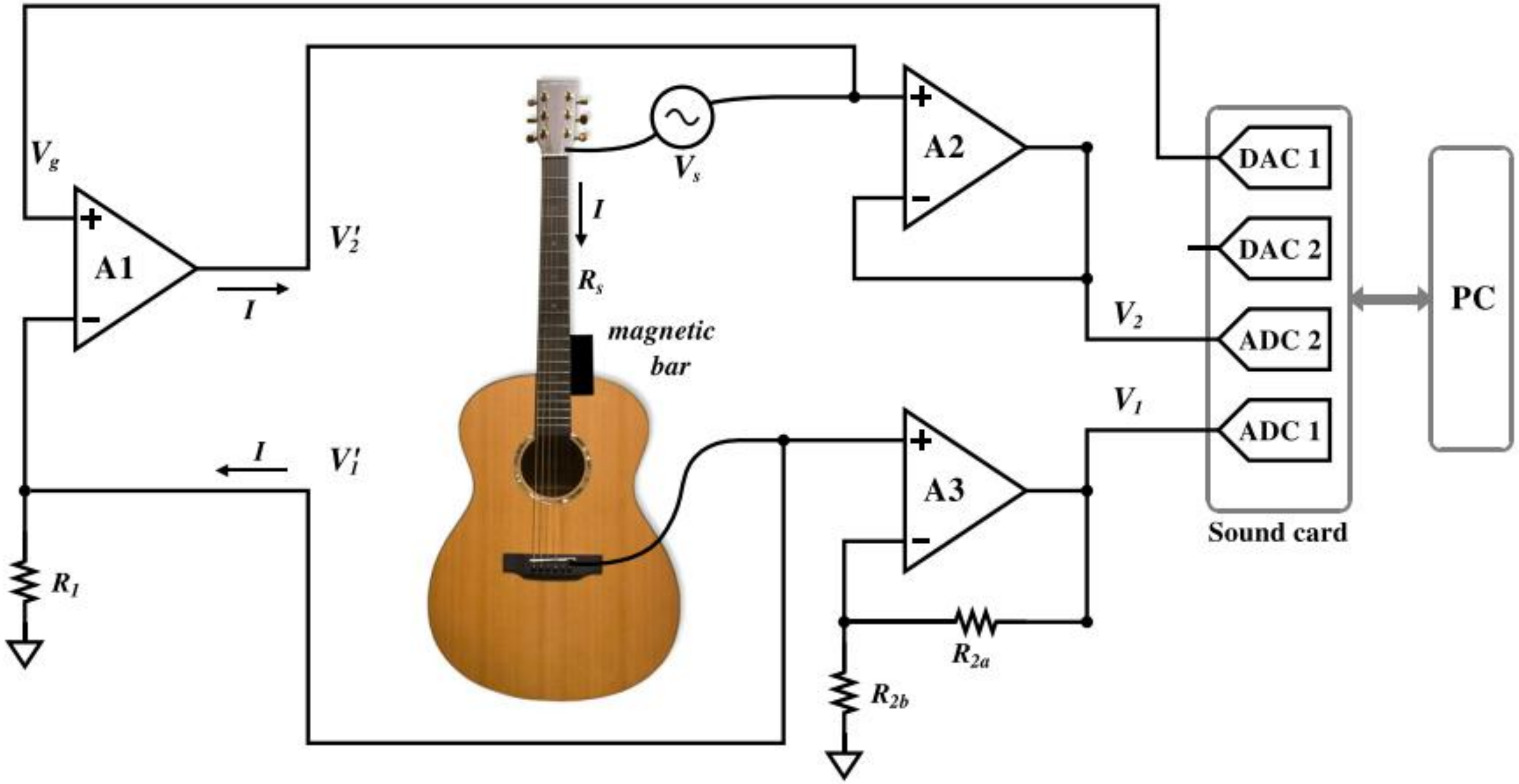
Setup – detection 2

- Either use very high-speed camera (slow motion),
- or capture a large amount of photos with high shutter speed
  - then pick one with highest amplitude,
- or take photos with slow enough shutter speeds
  - ~ shutter speed (time) $\gtrsim$ 1/frequency of vibrations to capture a whole period
  - motion-blurred, but maxima visible

M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

Setup – detection 3

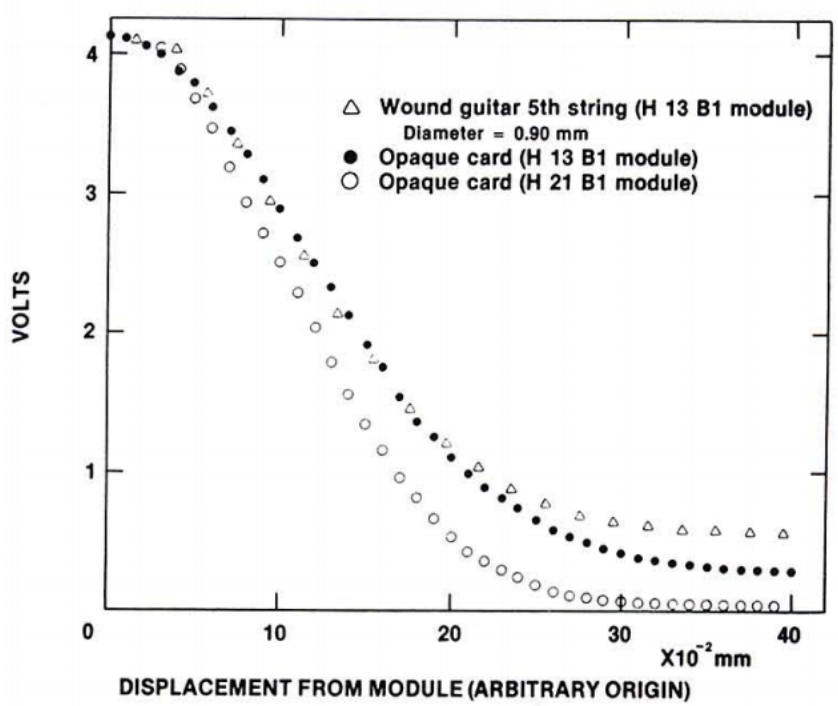
- Accurate, but only measures amplitude in 1 t. direction

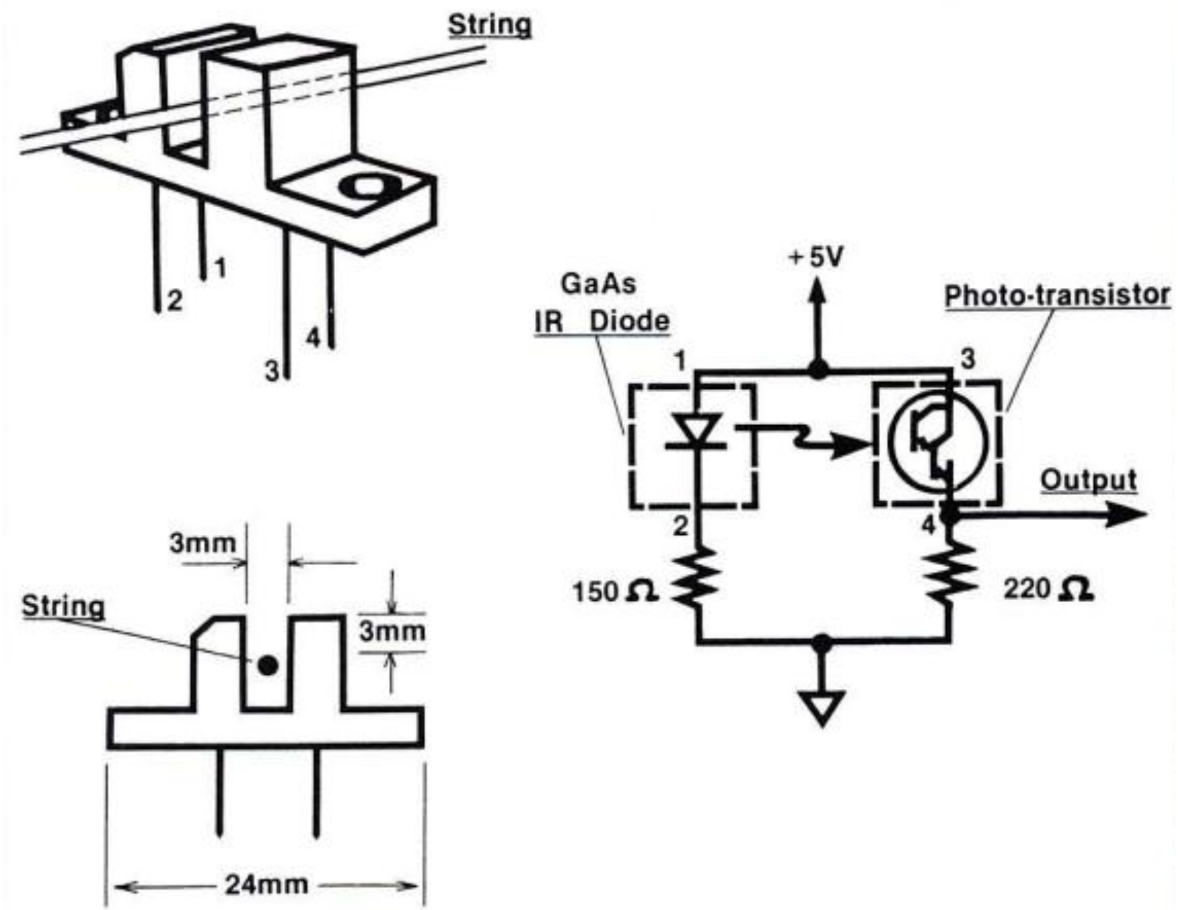


M. Carla, Measurements on a guitar string as an example of a physical nonlinear driven oscillator. AJP (2017)

Setup – detection 4

- can have 2 of these,
  - measure amplitude in both transverse





R. J. Hanson, Optoelectric Detection for String Vibration. The Physics Teacher (1987); doi: 10.1119/1.2342200

Conclusion

What you should work on

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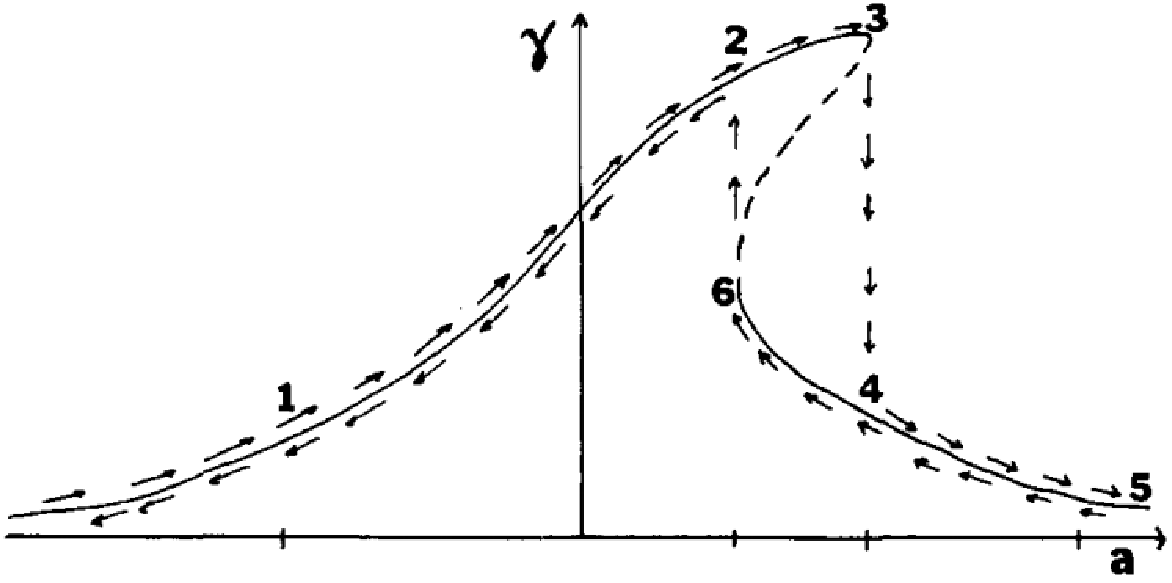
To measure: Duffing response curve



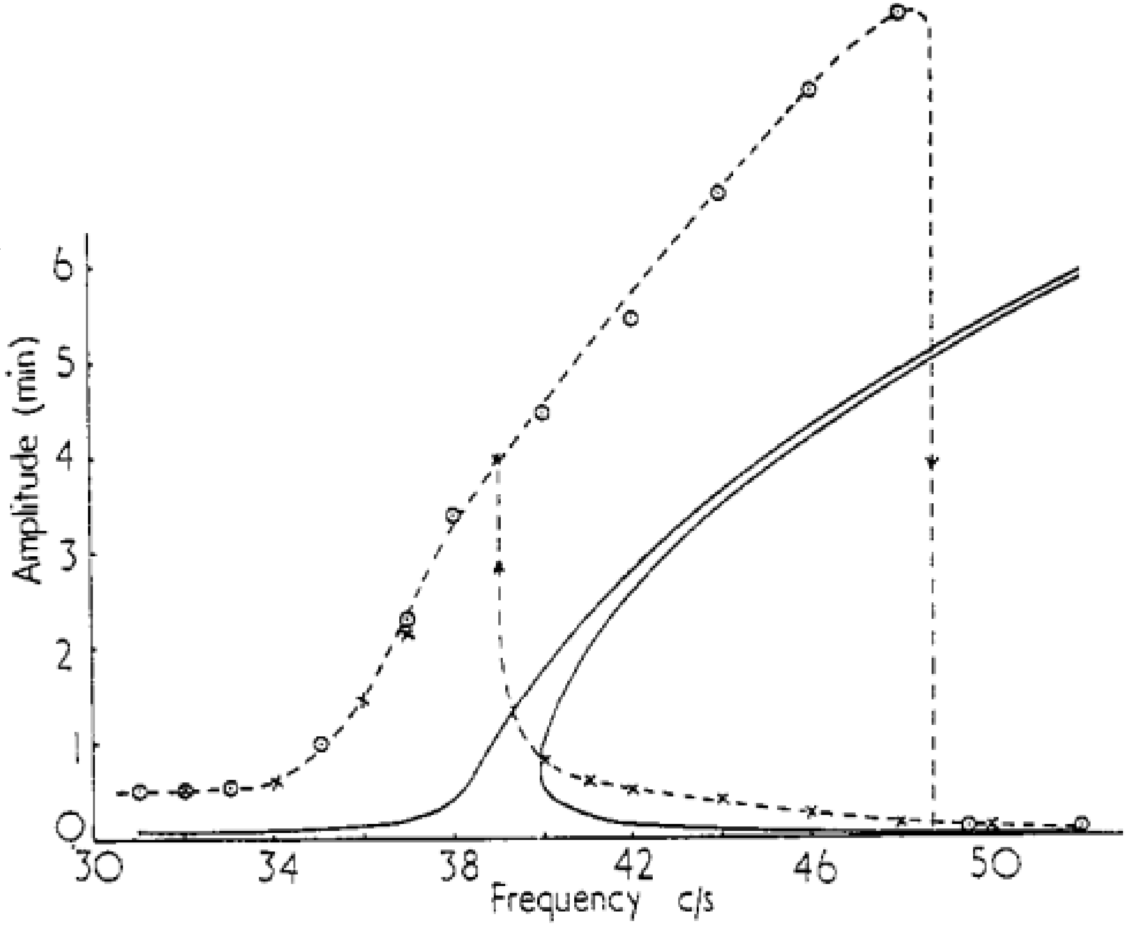
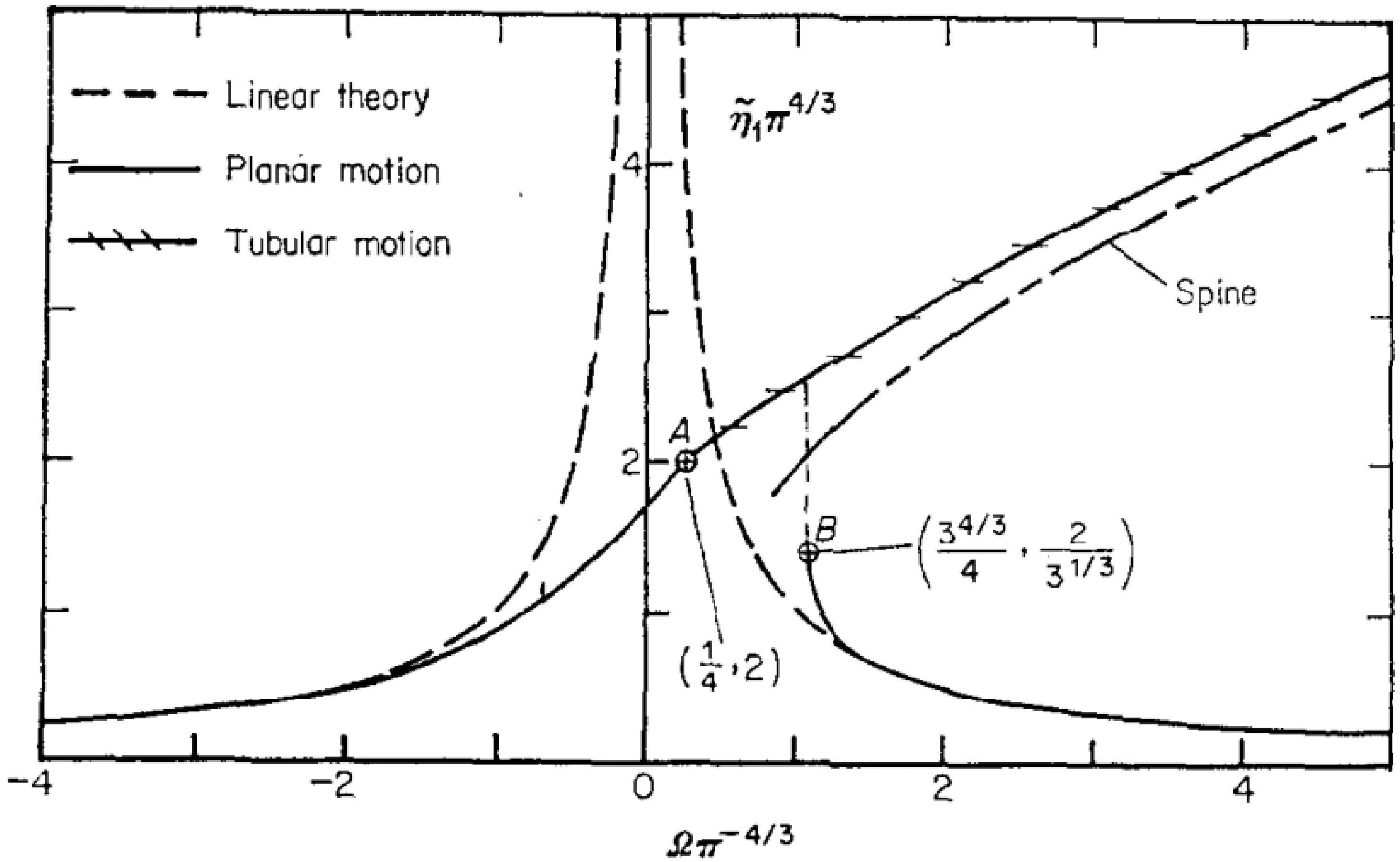


Fig. 2. Amplitude-frequency characteristics of string in its fundamental mode

——— calculated from equation (5)
— — — experimental points
○ ○ ○ frequency increasing
× × × frequency decreasing

N. B. Tufillaro, Nonlinear and chaotic string vibrations. Am. J. Phys. 57, 408 (1989); doi: 10.1119/1.16011
E. W. Lee, Non-linear forced vibration of a stretched string. (1957) Br. J. Appl. Phys. 8 411

To measure: Non-planar instability



R. Narasimha Non-linear vibration of an elastic string. J. Sound Vib. (1968) 8 (1), 134-146

To try:
Non-harmonic forcing

Chaos?

- J.M. Johnson and A.K. Bajaj, Amplitude modulated and chaotic dynamics in resonant motion of strings. Journal of Sound and Vibration (1989) 128(1), 87- 107
- N.B. Tufillaro, Nonlinear and chaotic string vibrations. Am. J. Phys. 57, 408 (1989); doi: 10.1119/1.16011
- A.K. Bajaj and J.M. Johnson, On the amplitude dynamics and crisis in resonant motion of stretched strings (1991)
- T.C. Molteno and N.B. Tufillaro, An experimental investigation into the dynamics of a string, Am. J. Phys. 72, 1157 (2004); doi: 10.1119/1.1764557

- not much done experimentally….

Theory

- Understand some of the basics from the literature
- Compare different models to your experiments
- Find new approximations
    - preferably easier but still working

References

- STEM Fellowship
- Sources of videos:
  - Brotherof
  - Dan Russel
  - Prof Miller
  - SMUPhysics
- Pictures:
  - Theory
- A video series about string simulation
- Scientific papers cited in respective slides

Desmos waves superposition

https://www.desmos.com/calculator/365anbv6mq